Intermediate Heat Transfer
ME 6300

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Module 1
Introduction, Heat Conduction Equation
(Quick Summary Review of Undergraduate Material)

Lecture notes based on:

1. G. Nellis and S. A. Klein (2009), Heat Transfer, Cambridge University Press Lecture notes provided by Drs. Nellis and Klein

2. Other textbooks such as T.L. Bergman, A.S. Lavine, F.P. Incropera, D.P. DeWitt, "Fundamentals of Heat and Mass Transfer", 8th Edition

Modes of Heat Transfer

Heat transfer occurs whenever there is a temperature difference

Modes:

- <u>Conduction:</u> Stationary medium ΔT (interaction of microscale carriers within material)
- <u>Convection:</u> Heat transfer between a surface and a moving fluid (conduction with fluid motion, energy also transferred due to bulk fluid motion)
- <u>Radiation:</u> Heat transfer between two surfaces at different T even in absence of intervening medium, due to electromagnetic waves

Conduction

Energy transfer due to interactions of micro-scale energy carriers within material

- High energy (fast moving, high temperature) carriers collide with lower energy (slower moving, lower temperature) carriers
- Leads to transfer of energy from high T to low T

Type of energy carriers depends on material

- In gas or liquid, energy carriers are molecules
- In solid, energy carriers are phonons (lattice vibrations) or electrons

Regardless of the type of energy carriers, conduction heat transfer is characterized by ***Fourier's law***

Fourier's Law

Rate Equation

- Fourier's Law

$$\boxed{\dot{q}'' = -k\frac{\partial T}{\partial x}}$$

where:

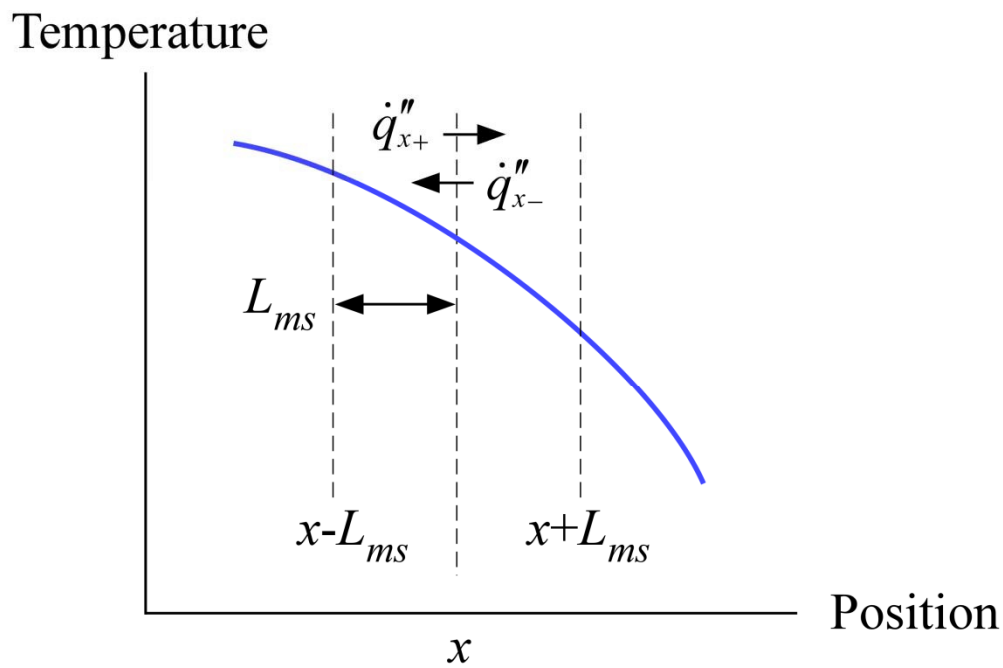
$\dot{q}''$ is the heat flux in the x-direction $\left(\mathrm{W/m^2}\right)$

k is the thermal conductivity $\left(\mathrm{W/m\text{-}K}\right)$

$\frac{\partial T}{\partial x}$ is the temperature gradient in the x-direction $\left(\mathrm{K/m}\right)$

- - because heat flows in direction of ↓T

Meaning of Thermal Conductivity



n_{ms} the number density of the energy carriers $\left(\#/\,\mathrm{m}^2\right)$

v_{ms} velocity of the energy carriers $(\mathrm{m/s})$

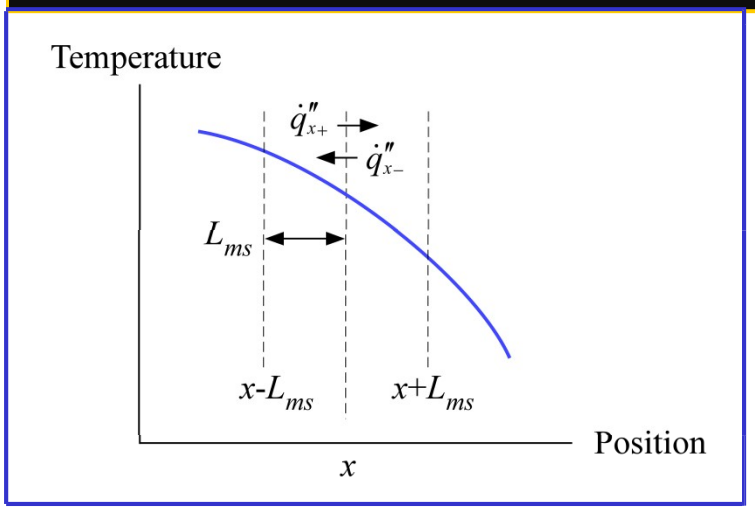
L_{ms} the average distance between energy carrier interactions (m)

c_{ms} the ratio of the energy of the carrier to its temperature (J/K)

energy transfer per area from left-to-right: $\dot{q}''_{x+} \approx \underbrace{n_{ms}\, v_{ms}}_{\frac{\#\text{ carriers}}{\text{area-time}}}\ \underbrace{c_{ms}\, T_{x-L_{ms}}}_{\frac{\text{energy}}{\text{carrier}}}$

energy transfer per area from right-to-left: $\dot{q}''_{x-} \approx \underbrace{n_{ms}\, v_{ms}}_{\frac{\#\text{ carriers}}{\text{area-time}}}\ \underbrace{c_{ms}\, T_{x+L_{ms}}}_{\frac{\text{energy}}{\text{carrier}}}$

Meaning of Thermal Conductivity



net energy transfer in the x-direction: $\dot{q}' \approx \dot{q}''_{x+} - \dot{q}''_{x-}$

$$\dot{q}'' \approx n_{ms} v_{ms} c_{ms} \left(T_{x-L_{ms}} - T_{x+L_{ms}} \right)$$

$$\dot{q}'' \approx -\underbrace{n_{ms} v_{ms} c_{ms} 2 L_{ms}}_{\text{thermal conductivity}} \underbrace{\frac{\left(T_{x+L_{ms}} - T_{x-L_{ms}} \right)}{2 L_{ms}}}_{\approx \frac{\partial T}{\partial x} = \lim_{dx \to 0} \frac{\left(T_{x+dx} - T_{x-dx} \right)}{2\,dx}} = -k \frac{\partial T}{\partial x}$$

- Provided that the **Knudsen number**

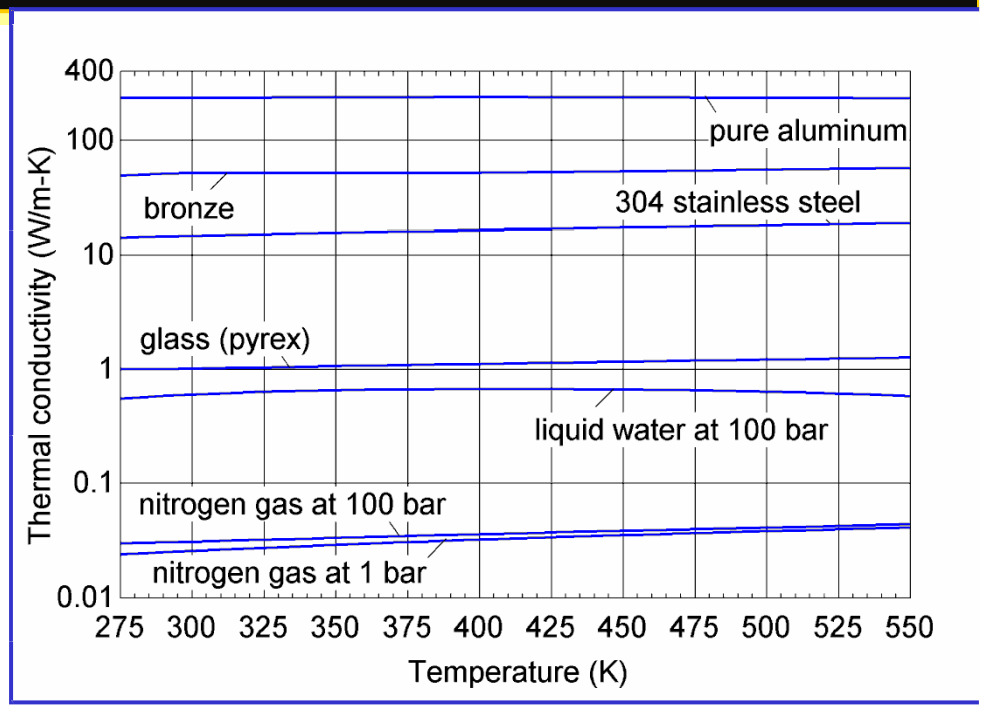
$$Kn = \frac{L_{ms}}{L_{char}}$$

- is small, Fourier's law holds and the thermal conductivity represents the product of the:
- number density (nms),
- average velocity (vms),
- heat capacity (cms), and
- distance between interactions (Lms).
- for the energy carriers in the material

$$k \propto n_{ms} v_{ms} c_{ms} L_{ms}$$

Thermal Conductivity – Representative Values

- pure metals have the largest k
  - electrons are numerous and fast
  - conductivity is related to electrical resistivity (Wiedemann-Franz law)
- alloys have lower k
  - impurities impede the motion of electrons
- non-metals have lower k
  - energy carried by phonons
- liquids have lower k
  - energy carried by molecules
- gases have lowest k
  - molecules are diffuse
  - kinetic theory to determine the thermal conductivity of a gas



Material scientists and physicists examine the underlying structure of a material in order to predict thermal conductivity (and other properties)

1-D Conduction (Energy Balance)

Consider a 1-D system

$\dot{q}$ = internal heat generation per unit volume

Energy conducted in left face (1) (Flux)

$$q_x = -kA\frac{\partial T}{\partial x} +$$

Heat Generated in element (2) (Source) =

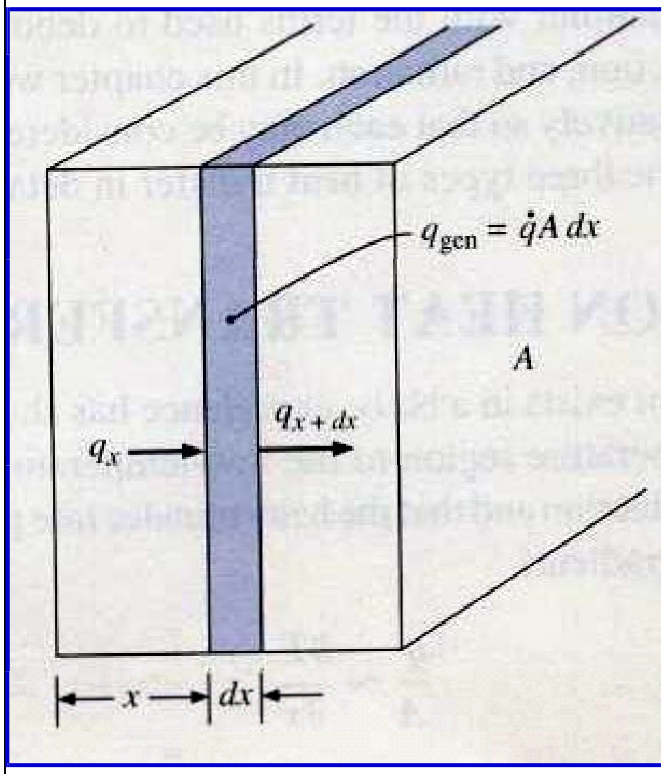
$$\dot{q}A\,dx$$

Change in internal energy (3) (Storage)

$$\rho cA\frac{\partial T}{\partial t}dx +$$

Energy conducted out right face (4) (Flux)

$$q_{x+dx} = -kA\frac{\partial T}{\partial x}\bigg|_{x+dx} = -A\left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]$$



Energy Balance

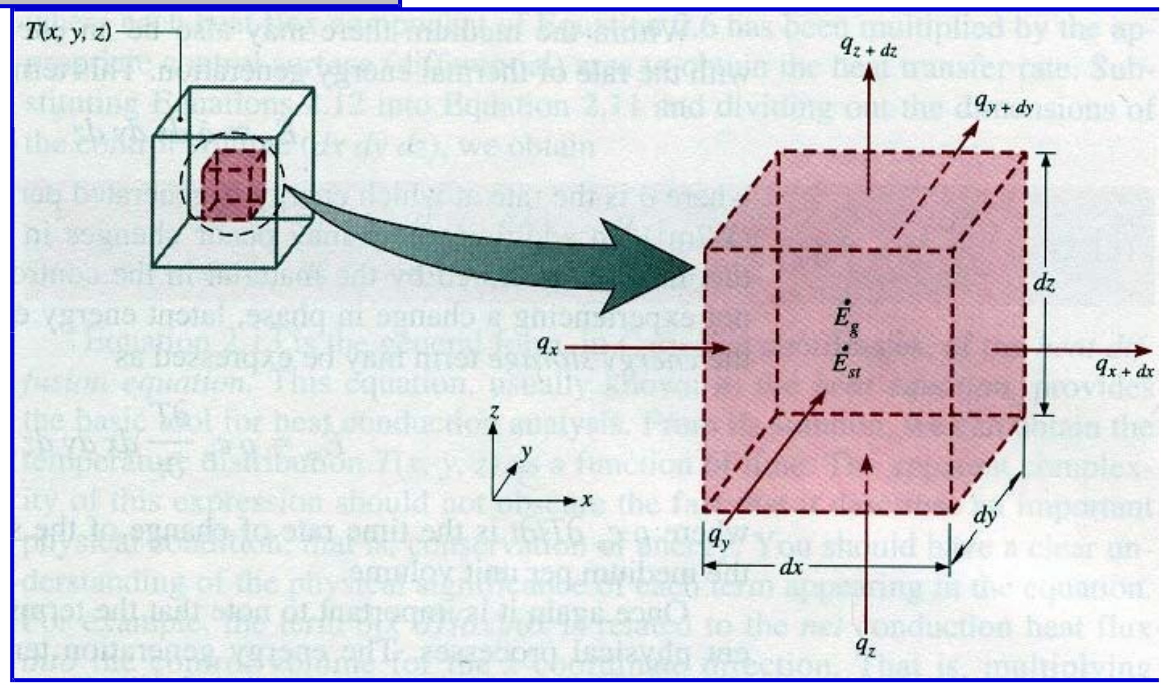
$$q_{x+dx} = -kA\left.\frac{\partial T}{\partial x}\right|_{x+dx} = -A\left[k\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dx\right]$$

Energy Balance

Heat Diffusion Equation

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\dot{q}=\rho c\frac{\partial T}{\partial t} \quad 1-D \quad \textbf{Heat Conduction Equation}$$

Extend to 3-D:



$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c\frac{\partial T}{\partial t}$$

Cylindrical Coordinates

Cylindrical coordinates:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Simplification 1 (Constant k):

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\alpha = \frac{k}{\rho C} = \text{thermal diffusivity} \quad \left(\frac{m^2}{sec}\right)$$

Larger $\alpha \Rightarrow$ faster heat diffusion into material

- k high $\Rightarrow$ faster energy transfer
- ρC low $\Rightarrow$ less energy needed to change T (low thermal capacity)

Simplifications

Special Case 1: steady state 1-D Heat Flow (No Heat Gen)

$$\frac{d^2T}{dx^2} = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Special Case 2: steady state 1-D Heat Flow (No Heat Gen)

Cylindrical

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = 0$$

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Special Case 3: steady state 1-D Heat Flow with Heat Gen

(Cartesian)

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

Special Case 4: 2-D steady state without Heat Gen

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Boundary and Initial Conditions

Solution to the heat diffusion equation depends on:

- Physical conditions at boundaries of the medium
- Conditions existing in the medium at some initial time (time dependent problems)

Heat Equation is 2nd order in space

$\Rightarrow$ 2 BC for <u>each coordinate</u> needed to describe system

1st order in time

$\Rightarrow$ 1 initial condition required

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Boundary and Initial Conditions

<u>Types of BC at Surface (x=0)</u>

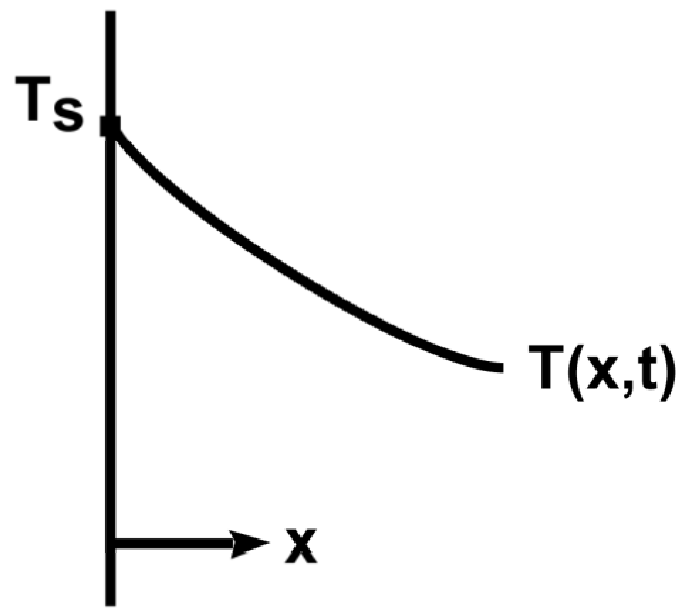
1. Surface maintained at constant temperature T_s

   $T(0,t) = T_s$

   B.C of the first kind (Dirichlet condition)

   e.g., when surface is in contact with boiling liquid

   $T_s = T_{phase\ change}$



Boundary and Initial Conditions

2. <u>Constant surface heat flux</u>

a. Finite heat flux

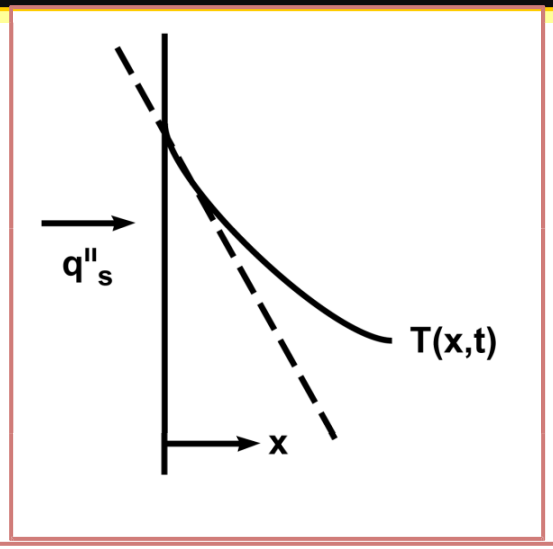
$$\boxed{-k\left.\frac{\partial T}{\partial x}\right|_0 = q_s''}$$

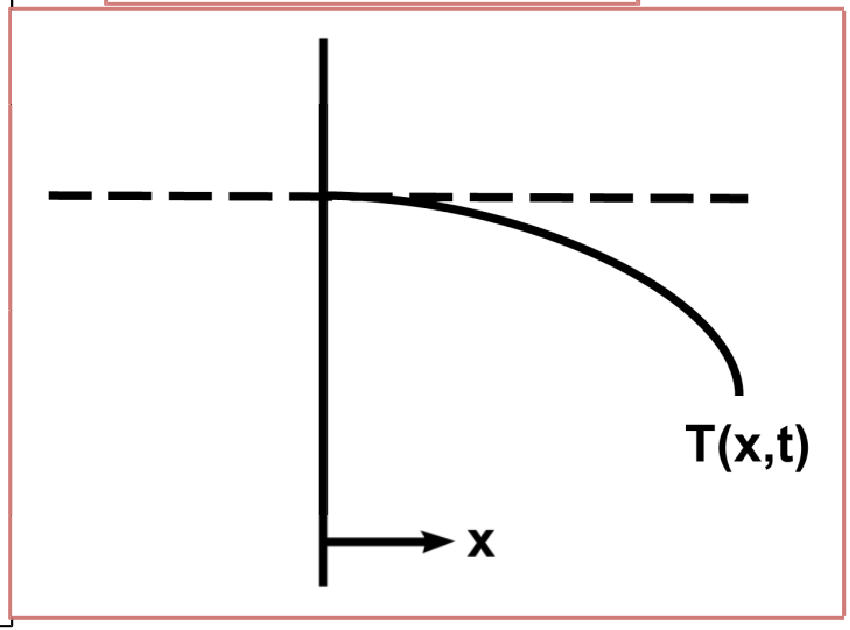
Neumann condition (B.C. of 2nd kind)

e.g., electric heater

b. Adiabatic (insulated) surface

$$\boxed{\left.\frac{\partial T}{\partial x}\right|_o = 0}$$



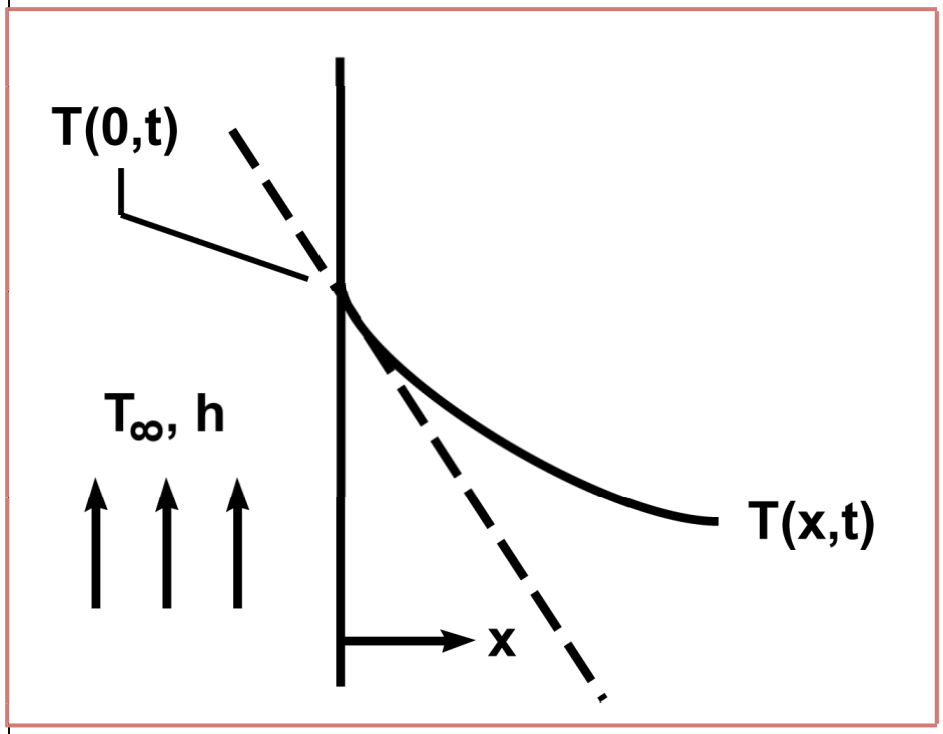


Boundary and Initial Conditions

3.Convection Surface Condition

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = h\left(T_\infty - T_{(0,t)}\right)$$

Mixed Boundary Condition



Plane Wall

- 1-D, steady-state (plane wall: *A* not f(*x*))

$$\dot{q} = -k\,A_c \frac{\partial T}{\partial x} \quad \text{(Fourier's law)}$$

$$\frac{d^2T}{dx^2} = 0$$

- Upon integration and application of BC:

$$\frac{d^2T}{dx^2} \text{ with } T_{x=0} = T_H \text{ and } T_{x=L} = T_C$$

- Temperature profile:

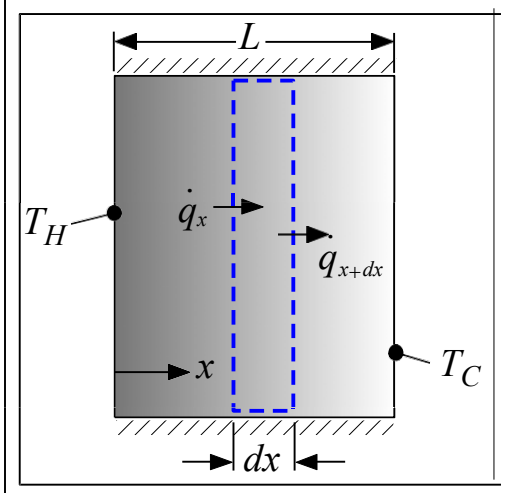
$$T = \frac{(T_C - T_H)}{L} x + T_H$$

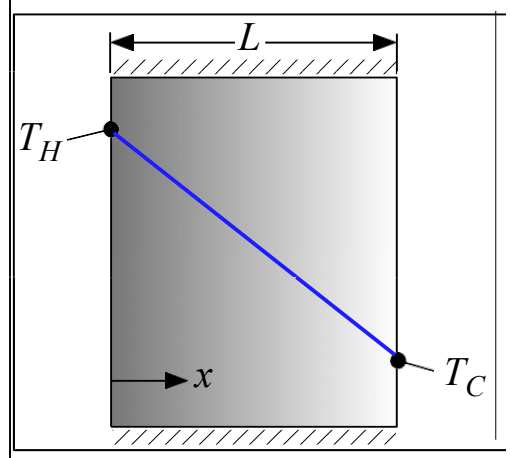
- Heat transfer rate:

$$\dot{q} = -k\,A_c \frac{dT}{dx} \qquad \dot{q} = \frac{k\,A_c}{L}(T_H - T_C)$$

- Thermal Resistance:

$$R = \frac{(T_H - T_C)}{\dot{q}}; R = \frac{L}{kA_c}$$





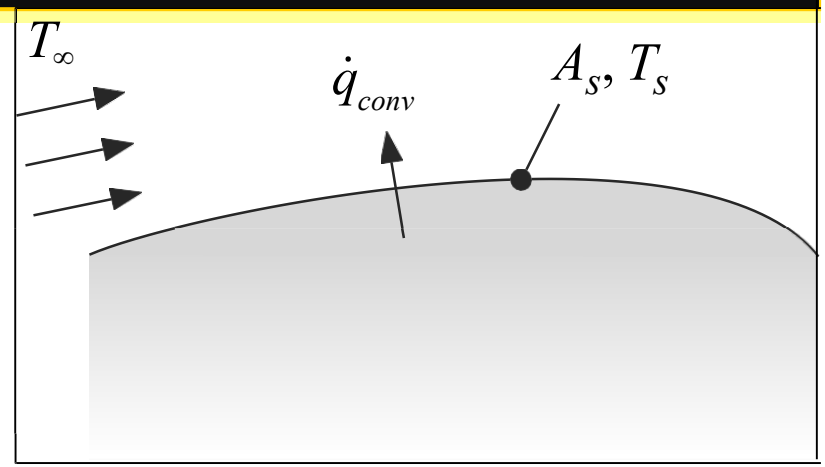
Convection Resistance

- *Convection* refers to heat transfer between surface and moving fluid
- *Newton's law of cooling* rate equation that characterizes convection

$$\boxed{\dot{q}_{conv} = \underbrace{\bar{h}}_{1/R_{conv}} A_s \left(T_s - T_\infty\right)}$$

  - *h* NOT material or fluid property
  - *h* complex function of geometry, fluid properties, and flow conditions
  - Chapters 4 through 7 of text book dedicated to determining *h*

$$\boxed{R_{conv} = \frac{1}{\bar{h}\, A_s}}$$



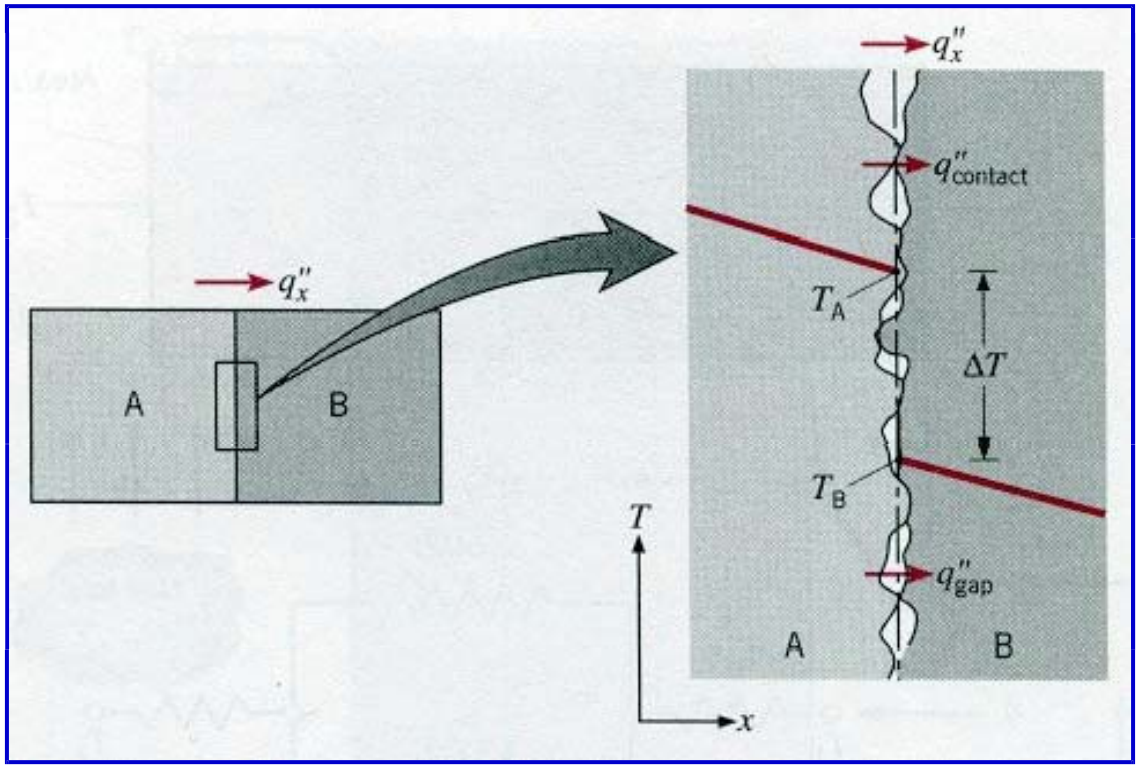
$\dot{q}_{conv}$	convection heat transfer rate (W)
$\bar{h}$	average heat transfer coefficient (W/m^2-K)
A_s	surface area exposed to fluid (m^2)
T_s	surface temperature (K)
T_∞	fluid temperature (K)

Contact Resistance

- Temperature drop across interface between materials may be appreciable
  - Thermal contact resistance $R_{t,c}$
  - For a unit area of the interface

$$R_{t,c}'' = \frac{T_A - T_B}{q_x''}$$

- Primarily due to surface roughness, gaps are typically air-filled
- Depends on: materials, surface preparation, interstitial material, clamping pressure
- Often ***THE*** dominant resistance in the entire system, esp. for high-flux applications



Contact Resistance

Heat Transfer = q through actual contact area (conduction)

+ q through conduction and radiation across gaps

Contact resistance = Parallel combination of contact spots and gaps

usually contact area is small

For solids with $k > k_{\text{interfacial fluid}}$, contact resistance reduced by:

- increasing area of contact spots through increased contact pressure
- reducing interface roughness
- introducing interfacial fluid of high k

Contact between dissimilar solids/interstitial filler materials

Typical filler material:

- soft metals
- thermal gasses

Other permanent interfaces: epoxy, soft solder, etc.

Contact Resistance

- *Area-specific contact resistance* typically measured and values for specific combinations are available in tables and handbooks (Table 1-1)
  - only meant to provide estimate of importance for typical applications

Table 1-1: Area-specific contact resistance for some interfaces, from Schneider (1985) and Fried (1969)

Materials	Clamping pressure	Surface roughness	Interstitial material	Temperature	Area-specific contact resistance
copper-to-copper	100 kPa	0.2 μm	vacuum	46°C	1.5×10^{-4} K-m^2/W
copper-to-copper	1000 kPa	0.2 μm	vacuum	46°C	1.3×10^{-4} K-m^2/W
aluminum-to-aluminum	100 kPa	0.3 μm	vacuum	46°C	2.5×10^{-3} K-m^2/W
aluminum-to-aluminum	100 kPa	1.5 μm	vacuum	46°C	3.3×10^{-3} K-m^2/W
stainless-to-stainless	100 kPa	1.3 μm	vacuum	30°C	4.5×10^{-3} K-m^2/W
stainless-to-stainless	1000 kPa	1.3 μm	vacuum	30°C	2.4×10^{-3} K-m^2/W
stainless-to-stainless	100 kPa	0.3 μm	vacuum	30°C	2.9×10^{-3} K-m^2/W
stainless-to-stainless	1000 kPa	0.3 μm	vacuum	30°C	7.7×10^{-4} K-m^2/W
stainless-to-aluminum	100 kPa	1.2 μm	air	93°C	3.3×10^{-4} K-m^2/W
aluminum-to-aluminum	1000 kPa	0.3 μm	air	93°C	6.7×10^{-5} K-m^2/W
aluminum-to-aluminum	100 kPa	10 μm	air	20°C	2.8×10^{-4} K-m^2/W
aluminum-to-aluminum	100 kPa	10 μm	helium	20°C	1.1×10^{-4} K-m^2/W
aluminum-to-aluminum	100 kPa	10 μm	hydrogen	20°C	0.72×10^{-4} K-m^2/W
aluminum-to-aluminum	100 kPa	10 μm	silicone oil	20°C	0.53×10^{-4} K-m^2/W

Radiation Resistance (Linearization)

- The radiation rate equation can be rearranged so that it resembles a resistance equation

$$\dot{q}_{rad} = A_s \sigma \varepsilon \left(T_s^4 - T_{sur}^4 \right)$$

$$\dot{q}_{rad} = \underbrace{A_s \sigma \varepsilon \left(T_s^2 + T_{sur}^2 \right) \left(T_s + T_{sur} \right)}_{1/R_{rad}} \left(T_s - T_{sur} \right)$$

- Radiation resistance (exact):

$$\boxed{R_{rad} = \frac{1}{A_s \sigma \varepsilon \left(T_s^2 + T_{sur}^2 \right) \left(T_s + T_{sur} \right)}}$$

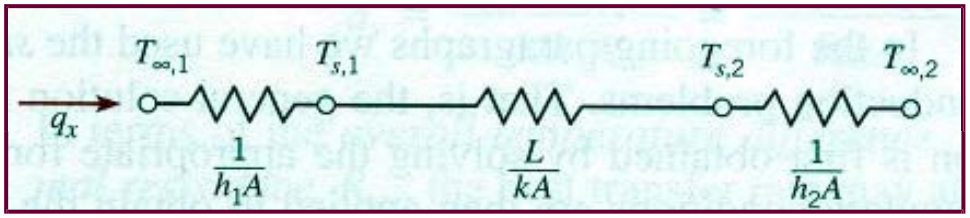
- Radiation resistance (approximate):

$$R_{rad} \approx \frac{1}{A_s \sigma \varepsilon \, 4 \bar{T}^3} \quad \text{where } \bar{T} = \frac{T_s + T_{sur}}{2}$$

  – works well if the absolute temperatures of the surface and surroundings are both large and not too different from each other

Resistance Networks

- Equivalent thermal circuit:



$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

- Heat transfer (current) same through each resistor. Temperature (voltage) drops proportional to resistances
- Total thermal resistance: $\frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}} = q_x$
- Series resistances can be summed: $R_{tot} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$

Resistance Networks

- Composite Wall:

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R}$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\frac{1}{h_1 A} + \frac{L_A}{k_A A} + \frac{L_B}{k_B A} + \frac{L_C}{k_C A} + \frac{1}{h_4 A}}$$

- Can also write equations for individual resistances:

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{h_1 A} = \frac{T_{s,1} - T_2}{L_A / k_A A} = \frac{T_2 - T_3}{L_B / k_B A} = \ldots\ldots\ldots\ldots\ldots$$

- Define an overall heat transfer coefficient U:

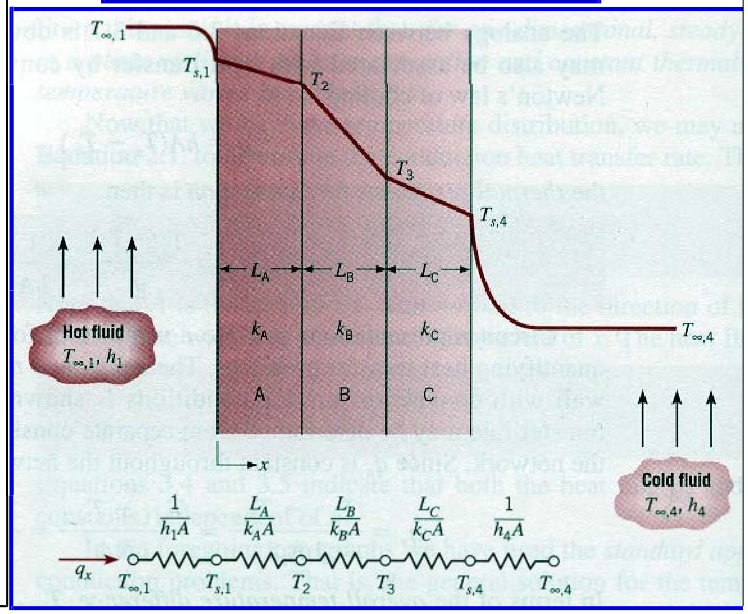
$$q_x = (UA)\Delta T$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t}$$

$$UA = \frac{1}{\sum R_t}$$

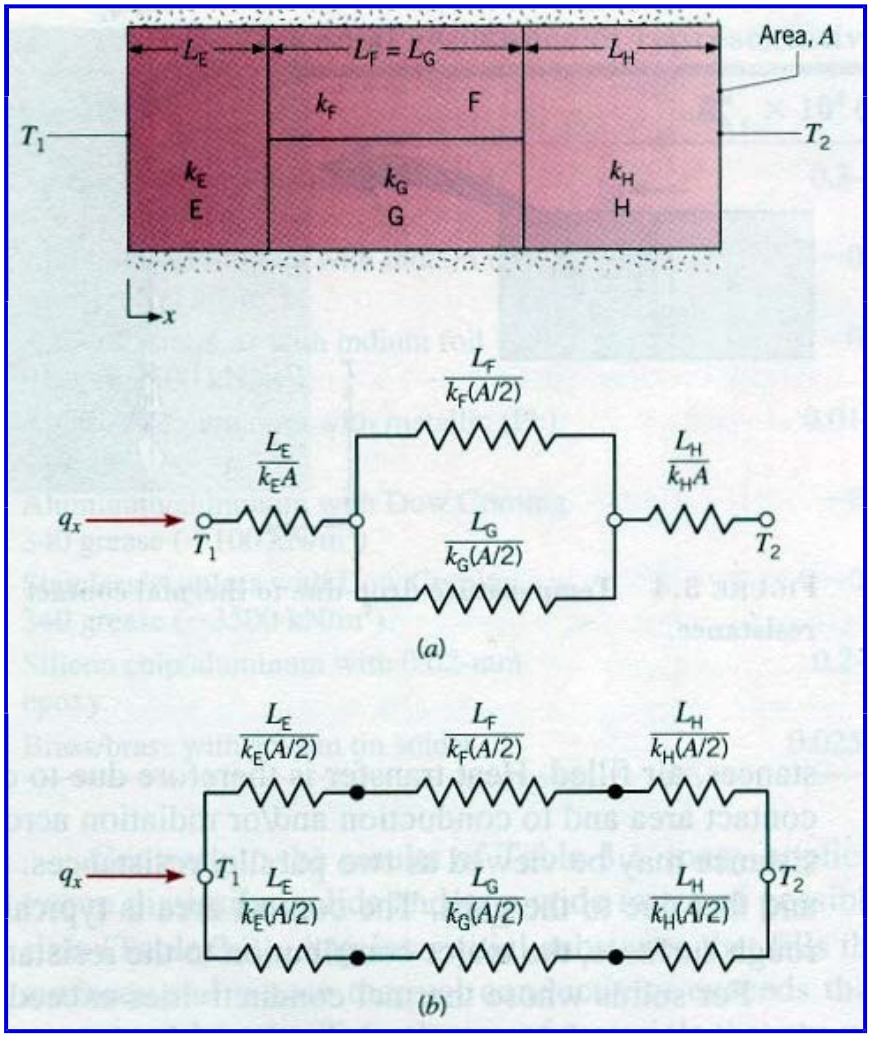
$$U = \frac{1}{R_{tot} A} = \frac{1}{\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_4}}$$

$$R_{tot} = \sum R_t = \frac{\Delta T}{q} = \frac{1}{UA}$$



Series/Parallel Configurations

- For the sake of completeness:



Radial Systems

Radial Systems

- Frequently, temperature gradients only in radial direction, so can treat as 1-D systems

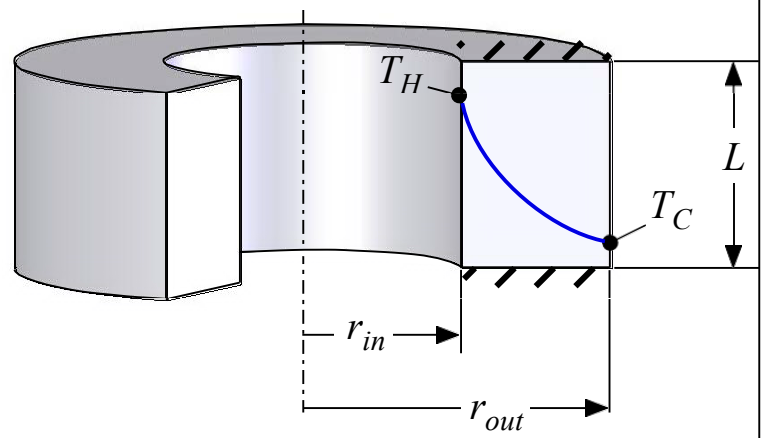
$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)=0$$

- Fourier's law:

$$q_r=-kA\frac{dT}{dr}=-k(2\pi rL)\frac{dT}{dr} \qquad A=2\pi rL$$

- q_r constant, q_r'' not, $A = f(r)$
- Integration yields: $T=c_1\ln r+c_2$
- Apply BC: $\dfrac{T-T_{s,out}}{T_{s,in}-T_{s,out}}=\dfrac{\ln(r/r_{out})}{\ln(r_{in}/r_{out})}$

$$q_r=q''A=-k2\pi rL\frac{dT}{dr}=\frac{2\pi Lk\,T_{s,in}-T_{s,out}}{\ln(r_{out}/r_{in})}$$



$$R_{cyl}=\frac{\ln\left(\frac{r_{out}}{r_{in}}\right)}{2\pi Lk}$$

T profile not linear despite no Q_{gen}, because A_s changes radially

Radial Systems

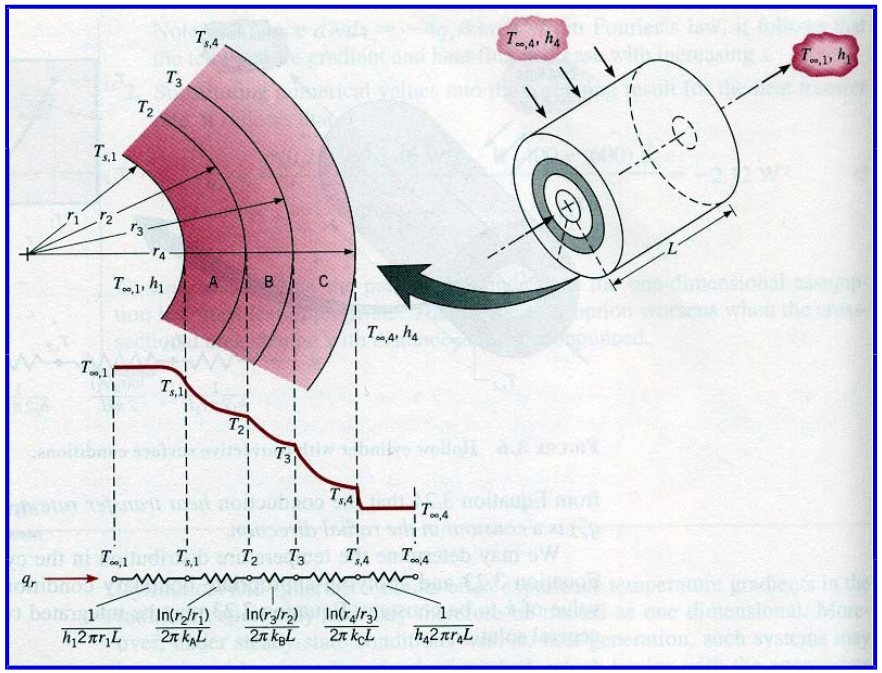
Composite Cylindrical Wall

- Use definition of resistance of cylindrical element for each cylinder (combine in series):

$$q_r = \frac{(T_{\infty 1} - T_{\infty 4})}{\frac{1}{2\pi r_1 L h_1} + \frac{\ln(r_2/r_1)}{2\pi k_A L} + \frac{\ln(r_3/r_2)}{2\pi k_B L} + \frac{\ln(r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}}$$

$$q_r = \frac{(T_{\infty 1} - T_{\infty 4})}{R_{tot}} = U_1 A_1 (T_{\infty 1} - T_{\infty 4})$$

$$A_1 = 2\pi r_1 L$$



U and UA of Composite Cylindrical Wall

- In plane wall systems, area A is unique and constant. But in cylindrical systems, choice of several surfaces evaluated at radius r_1, r_2, or r_3. But Q does not change because of choice of radius for evaluation of surface area

$$U_1 = \frac{1}{R_{tot} A_1}$$

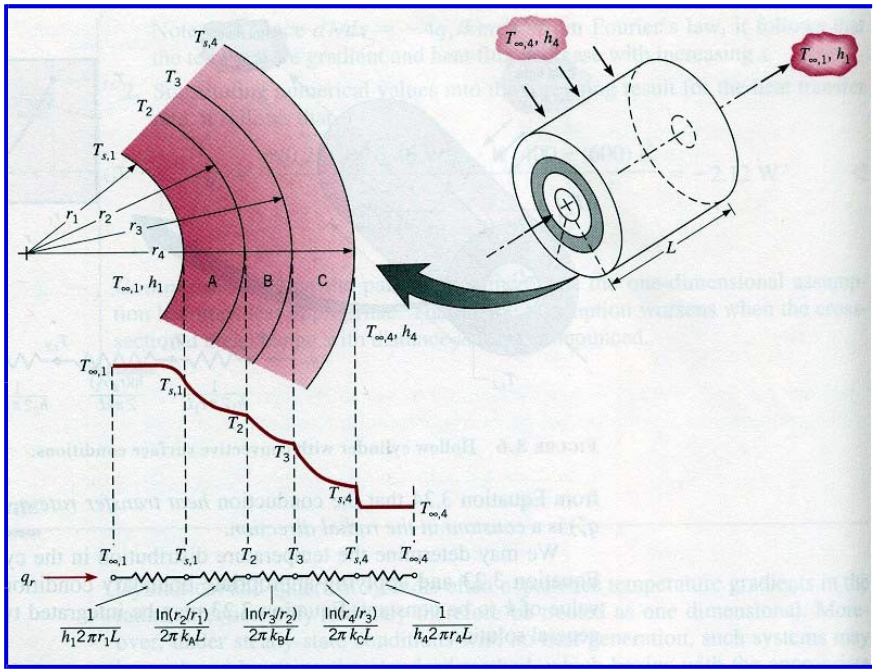
$$A_1 = 2\pi r_1 L$$

$$U_1 = \frac{1}{\frac{1}{h_1} + \ln(r_2/r_1)\frac{r_1}{k_A} + \frac{r_1}{k_B}\ln(r_3/r_2) + \frac{r_1}{k_C}\ln(r_4/r_3) + \frac{r_1}{r_4}\frac{1}{h_4}}$$

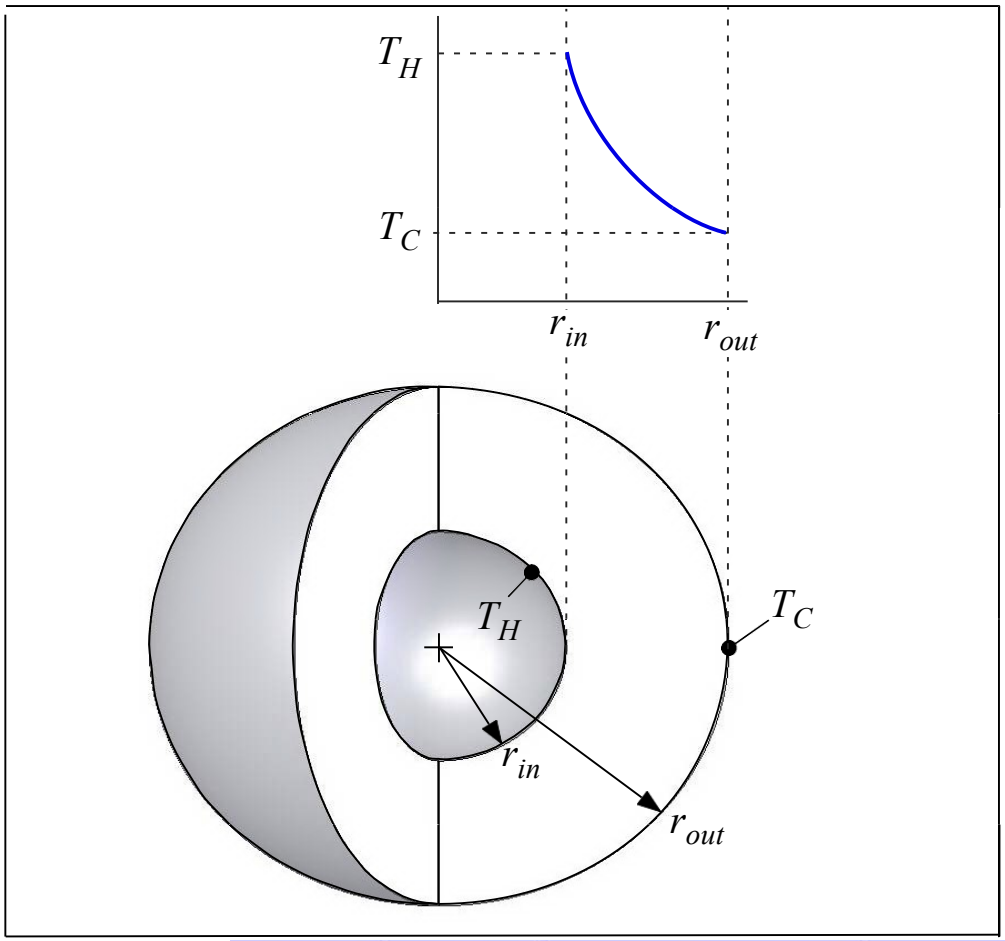
- U_1 defined from Inside Surface Area A_1
- Choice of reference surface area is arbitrary and/or based on personal preference and convenience

$$U_1 A_1 = U_2 A_2 = U_3 A_3 = U_4 A_4 = \left(\sum R_t\right)^{-1}$$

$$q_r = \frac{(T_{\infty 1} - T_{\infty 4})}{\frac{1}{2\pi r_1 L h_1} + \frac{\ln(r_2/r_1)}{2\pi k_A L} + \frac{\ln(r_3/r_2)}{2\pi k_B L} + \frac{\ln(r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}}$$

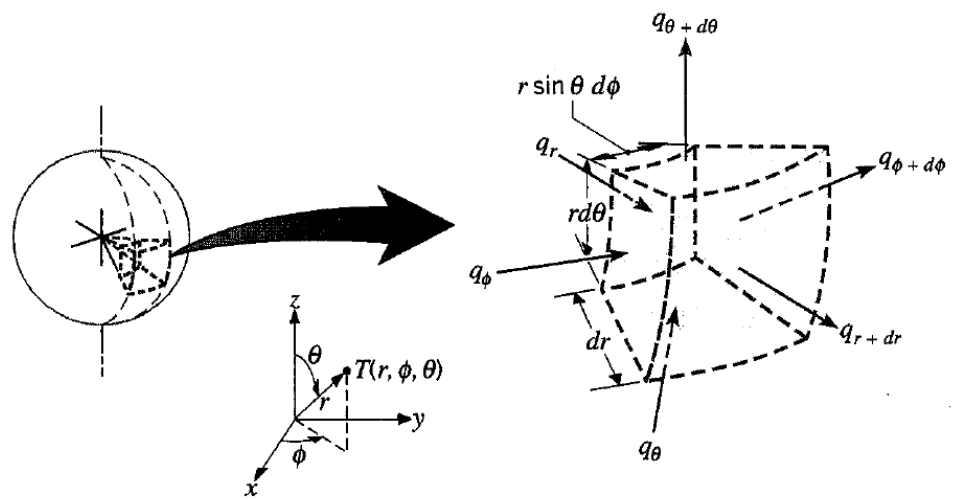


Spherical System



spherical shell:

$$R_{sph} = \frac{\left(\frac{1}{r_{in}} - \frac{1}{r_{out}}\right)}{4\pi k}$$



$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(k\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right) + \dot{q} = \rho c_p\frac{\partial T}{\partial t}$$

Resistance Formulae

Table 1-2: A summary of common resistance formulae

Situation	Resistance formula	Nomenclature
Plane wall	$R_{pw} = \dfrac{L}{k\,A_c}$	L = wall thickness (| | to heat flow) k = conductivity A_c = cross-sectional area ($\perp$ to heat flow)
Cylinder (radial heat transfer)	$R_{cyl} = \dfrac{\ln\left(\dfrac{r_{out}}{r_{in}}\right)}{2\pi L k}$	L = cylinder length k = conductivity r_{in} and r_{out} = inner and outer radii
Sphere (radial heat transfer)	$R_{sph} = \dfrac{1}{4\pi k}\left[\dfrac{1}{r_{in}} - \dfrac{1}{r_{out}}\right]$	k = conductivity r_{in} and r_{out} = inner and outer radii
Convection	$R_{conv} = \dfrac{1}{\bar{h}\,A_s}$	$\bar{h}$ = average heat transfer coefficient A_s = surface area exposed to convection
Contact between surfaces	$R_c = \dfrac{R_c''}{A_s}$	R_c'' = area specific contact resistance A_s = surface area in contact
Radiation (exact)	$R_{rad} = \dfrac{1}{A_s\,\sigma\,\varepsilon\left(T_s^2 + T_{sur}^2\right)\left(T_s + T_{sur}\right)}$	A_s = radiating surface area σ = Stefan-Boltzmann constant ε = emissivity T_s = absolute surface temperature T_{sur} = absolute surroundings temperature
Radiation (approximate)	$R_{rad} \approx \dfrac{1}{A_s\,\sigma\,\varepsilon\,4\,\bar{T}^3}$	A_s = radiating surface area σ = Stefan-Boltzmann constant ε = emissivity $\bar{T}$ = average absolute temperature

Intermediate Heat Transfer
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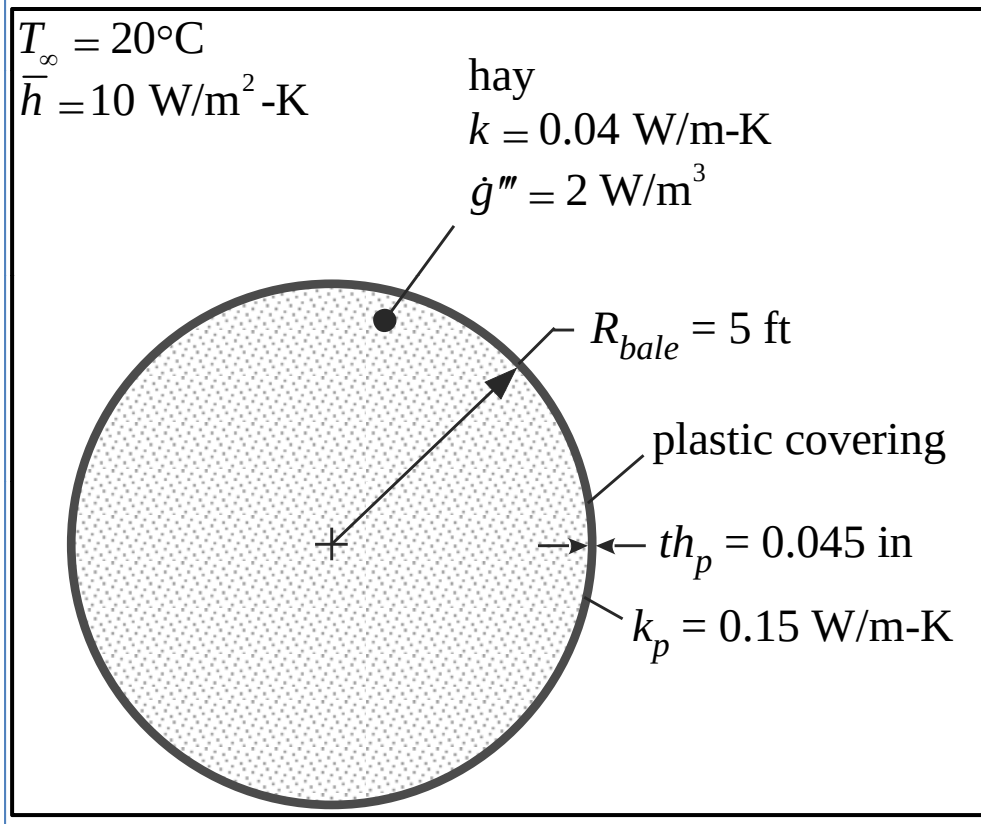
Module 3
Examples and Numerical Methods:
1-D Conduction with Heat Generation

1. G. Nellis and S. A. Klein (2009), Heat Transfer, Cambridge University Press Lecture notes provided by Drs. Nellis and Klein

2. Other textbooks such as T.L. Bergman, A.S. Lavine, F.P. Incropera, D.P. DeWitt, "Fundamentals of Heat and Mass Transfer", 8th Edition

Spontaneous Heating of a Hay Bale

- Thermal energy generation due to bacterial fermentation
  - Volumetric generation depends on moisture content of hay when baled
  - Bale too large and/or too wet - spontaneous combustion when T > 70 C



Governing Equation

- Energy balance:

$$\dot{q}_r + \dot{g} = \dot{q}_{r+dr}$$

$$\cancel{\dot{q}_r} + \dot{g} = \cancel{\dot{q}_r} + \frac{d\dot{q}}{dr}dr$$

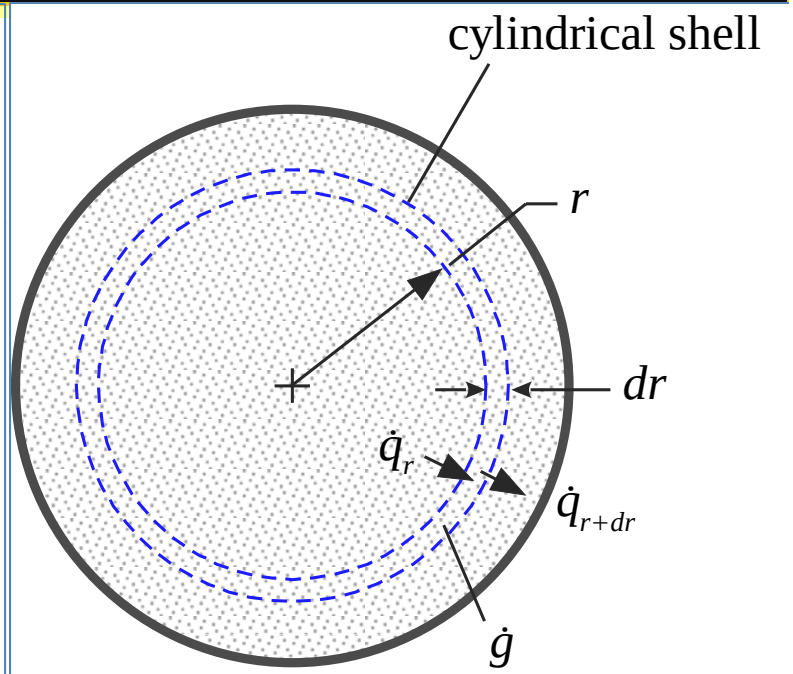
$$\boxed{\dot{g} = \frac{d\dot{q}}{dr}dr}$$

- Heat to be conducted out ↑ as r ↑ (generation)
- With rate equations:

$$\underbrace{\cancel{2\pi}\, r\, \cancel{L}\cancel{dr}\, \dot{g}'''}_{\dot{g}\text{ inside control volume}} = \frac{d}{dr}\left[\underbrace{-\cancel{2\pi}\, r\, \cancel{L}\, k\frac{dT}{dr}}_{\dot{q}\text{ in } r\text{-direction}}\right]\cancel{dr}$$

$$\boxed{r\dot{g}''' = -k\frac{d}{dr}\left(r\frac{dT}{dr}\right)}$$

- linear, 2nd-order, non-homogeneous, ODE



Solution

- Separate and integrate:

$$\int d\left(r\frac{dT}{dr}\right) = -\int \frac{r\dot{g}'''}{k}dr$$

$$r\frac{dT}{dr} = -\frac{r^2\dot{g}'''}{2k} + C_1$$

- Again to get general solution:

$$\int dT = \int\left(-\frac{r\dot{g}'''}{2k} + \frac{C_1}{r}\right)dr$$

$$\boxed{T = -\frac{r^2\dot{g}'''}{4k} + C_1\ln(r) + C_2}$$

- 2 BC (r = 0, r = R)
  - r = 0: symmetry, 0 energy generation

$$\dot{q}_{r=0} = 0 \qquad \left(-2\pi r L k\frac{dT}{dr}\right)_{r=0} = 0$$

- BC @ r = 0 yields C1 = 0

$$\boxed{T = -\frac{r^2\dot{g}'''}{4k} + C_2}$$

- As in previous example, @ r = R, T not specified, but obtain from surface energy balance and T_∞
- Energy into plastic cover = energy convected out by convection

$$\dot{q}_{out} = \frac{T_{r=R_{bale}} - T_\infty}{R_{cond,p} + R_{conv}}$$

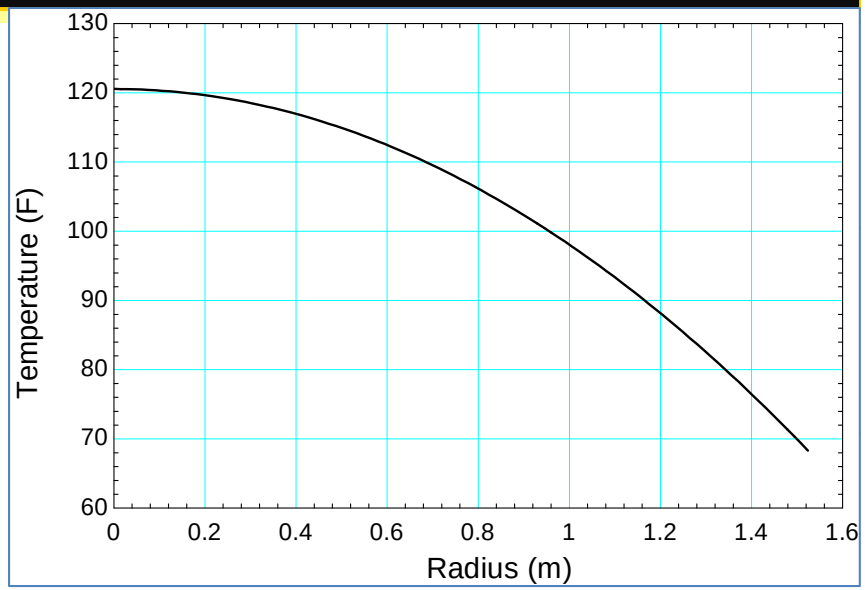
$$R_{cond,p} = \frac{\ln\left(\frac{R_{bale}+th_p}{R_{bale}}\right)}{2\pi L k_p} \approx \frac{th_p}{2\pi R_{bale} L k_p} \qquad R_{conv} = \frac{1}{2\pi\left(R_{bale}+th_p\right)L\bar{h}}$$

Solution

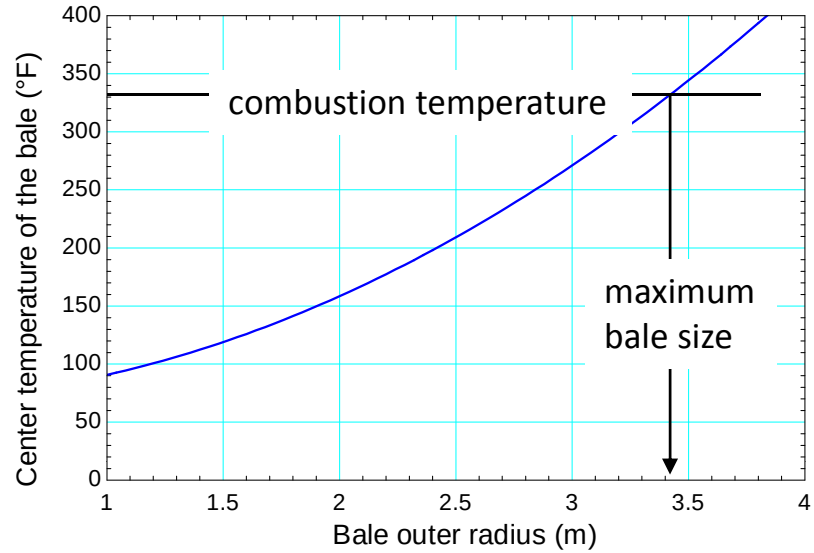
Unit Settings: [J]/[K]/[Pa]/[kg]/[radians]

C_2 = 322.3 [K]	$dTdr_{Rbale}$ = -38.1 [K/m]	$\dot{g}_v$ = 2 [W/m^3]
h = 10 [W/m^2-K]	k = 0.04 [W/m-K]	k_p = 0.15 [W/m-K]
L = 1 [m]	$\dot{q}_{out}$ = 14.59 [W]	$\dot{q}_{Rbale}$ = 14.59 [W]
R_{bale} = 1.524 [m]	$R_{cond,p}$ = 0.0007958 [K/W]	R_{conv} = 0.01044 [K/W]
th_p = 0.001143 [m]	T_∞ = 293.2 [K]	T_{Rbale} = 293.3 [K]

No unit problems were detected.



Parametric analysis on R to obtain limit for combustion



Nonuniform Heat Generation

- Let

$$\boxed{\begin{array}{c} \dot{g}''' = a + bT \\ \text{where} \\ a = -1\,\text{W/m}^3 \text{ and } b = 0.01\ \text{W/m}^3\text{-K} \end{array}}$$

$$\underbrace{\cancel{2\pi}\, r\, \cancel{L}\cancel{dr}\, \overbrace{(a+bT)}^{\dot{g}'''}}_{\dot{g}\text{ inside control volume}} = \frac{d}{dr}\left[\underbrace{-\cancel{2\pi}\, r\, \cancel{L}\, k\frac{dT}{dr}}_{\dot{q}\text{ in } r\text{-direction}}\right]\cancel{dr}$$

$$\boxed{r(a+bT) = -k\frac{d}{dr}\left(r\frac{dT}{dr}\right)}$$

- yields ODE that is not separable:

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{r(a+bT)}{k}$$

- Integration of RHS presents problems due to T in argument

$$\int d\left(r\frac{dT}{dr}\right) = -\int \frac{r(a+bT)}{k}dr$$

Solution

- Can solve it in software such as Maple:

Bessel function of the first kind

order

$$T = C_2 \boxed{\text{BesselJ}}\left(\boxed{0}, \sqrt{\frac{b}{k}}r\right) + C_1 \boxed{\text{BesselY}\left(0, \sqrt{\frac{b}{k}}r\right)} - \frac{a}{b}$$

0th-order Bessel function of the first kind

0th-order Bessel function of the second kind

- BC same as before; r = 0, $\left(\frac{dT}{dr}\right)_{r=0}$ is bounded

$$\frac{dT}{dr} = -C_2 \underbrace{\text{BesselJ}\left(1, \sqrt{\frac{b}{k}}r\right)}\sqrt{\frac{b}{k}} - C_1 \underbrace{\text{BesselY}\left(1, \sqrt{\frac{b}{k}}r\right)}\sqrt{\frac{b}{k}}$$

one of these terms must approach infinity as *r* approaches zero
this will eliminate either C_1 or C_2

Solution

$$\lim_{r\to 0}\frac{dT}{dr} = -C_2 \underbrace{\text{BesselJ}\left(1,\sqrt{\frac{b}{k}}0\right)}_{=0}\sqrt{\frac{b}{k}} - C_1 \underbrace{\text{BesselY}\left(1,\sqrt{\frac{b}{k}}0\right)}_{=-\infty}\sqrt{\frac{b}{k}}$$

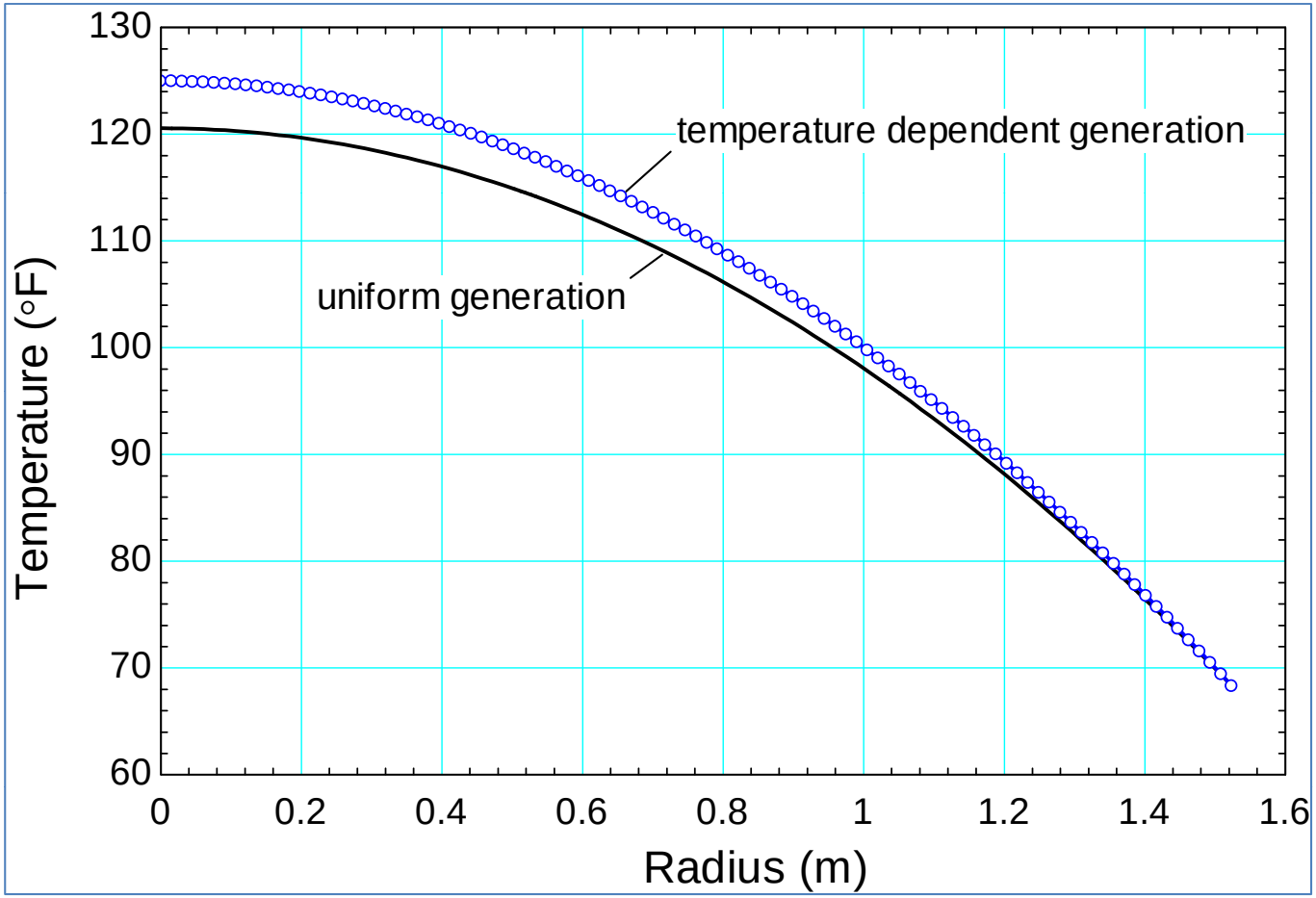
C_1 must be zero

$$\dot{q}_{r=R_{bale}} = \dot{q}_{out}$$

- At r = R_{bale}: $$-2\pi R_{bale} L k\left(\frac{dT}{dr}\right)_{r=R_{bale}} = \frac{\left(T_{r=R_{bale}} - T_\infty\right)}{R_{cond,p} + R_{conv}}$$
- One equation in unknown, C_2

Comparison of Solutions

- Note higher value of temperature-dependent solution – the bacterial fermentation heat source is larger at warmer temperatures



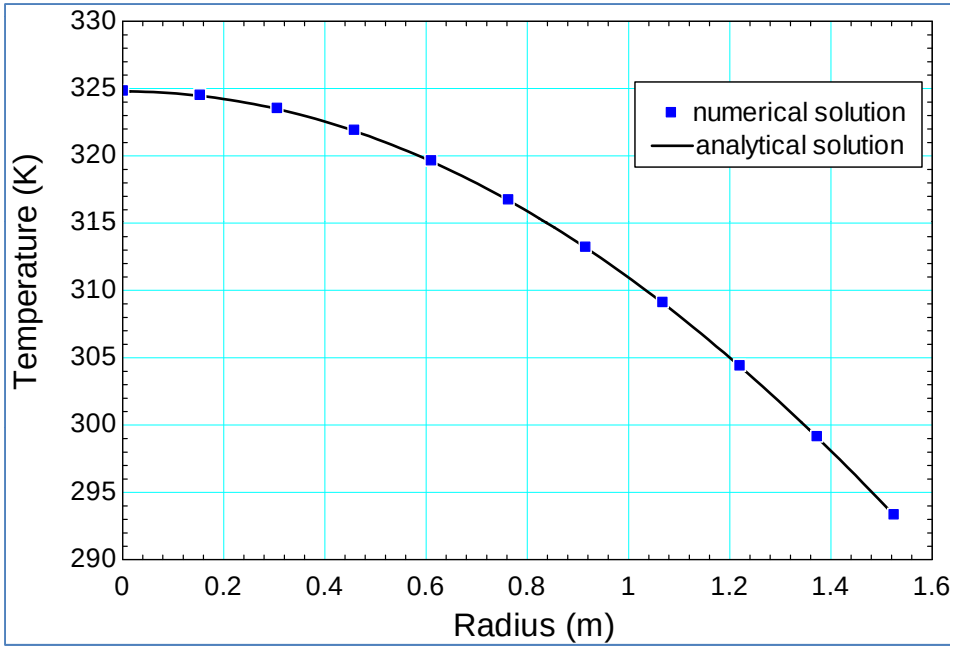
Numerical Solutions

Analytical solutions

- yield functions that satisfy governing equation over computational domain
  - exact solutions
  - computationally fast
  - inflexible - difficult (impossible) to include things like temperature-dependent properties

Numerical solutions

- predictions of temperature at discrete locations (nodes) within computational domain
  - approximate solutions
  - computationally slower
  - flexible - easy to include things like temperature-dependent properties



Approach

1. Define many (N) nodes (locations for T at Dx) and associated control volumes
2. Energy balance on each control volume
3. Approximate each energy term with rate equation (leads to approximation)
4. Solve using appropriate tool
5. Verify convergence
   a) enough nodes?
   b) examine dependence on N; should be invariant with N (tradeoff between accuracy and computational effort)
6. Check solution based on first principles, limits, overall energy balances, etc.
7. Compare numerical solution to corresponding analytical solution of simplified case

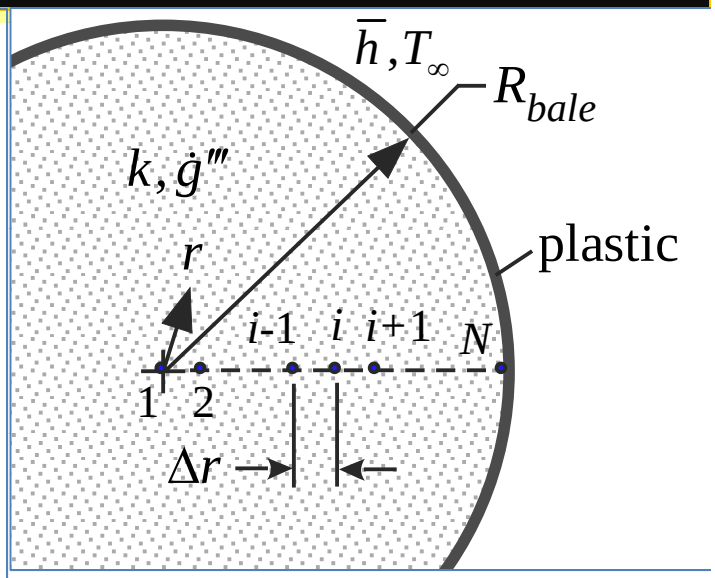
Don't be enamored of pretty pictures

Hay Bale revisited

- Uniformly distributed nodes:

$$r_i = \frac{(i-1)}{(N-1)} R_{bale} \quad \text{for } i = 1..N \qquad \Delta r = \frac{R_{bale}}{(N-1)}$$

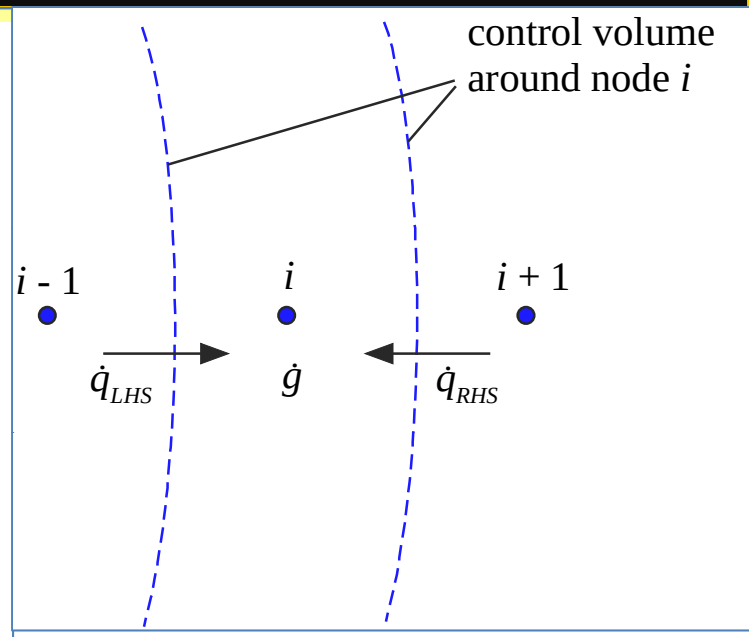
- In some cases, concentrate nodes in regions of interest (e.g., with large *dT/dr*)
- Use arrays
  Track variables common to all nodes (e.g., *r*, *T*)



Nodal Energy Balances

Internal Nodes:

- Energy balance exactly correct (energy balance must hold)
- Does not matter whether heat transfers assumed into or out of control volume (solution will indicate actual direction based on + or - values)

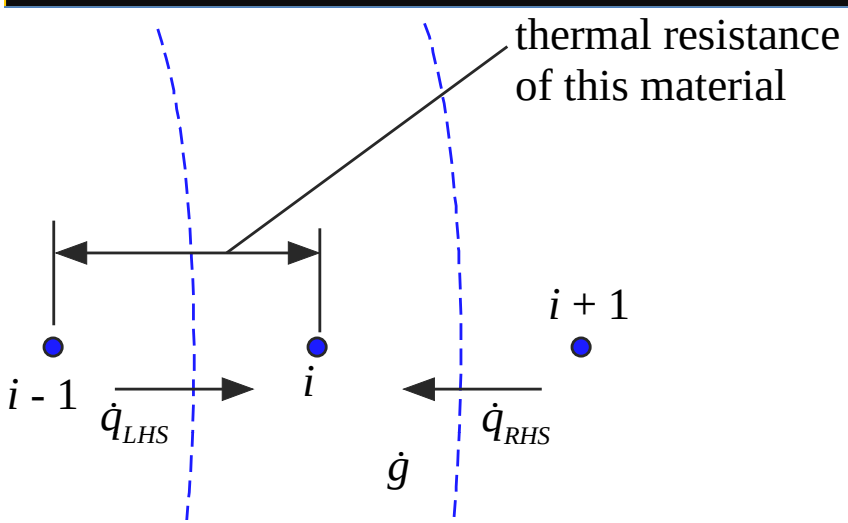


IN = OUT + STORED

$$\dot{q}_{LHS} + \dot{q}_{RHS} + \dot{g} = 0$$

defined as being into the control volume

Rate Equations



consistent with being into the control volume

$$\dot{q}_{LHS} \approx \frac{\boxed{\left(T_{i-1} - T_i\right)}}{\boxed{R_{LHS}}}$$

resistance of material separating nodes *i*-1 and *i*

Thermal resistance notes

- rate equation: $R_{LHS} = \dfrac{\boxed{\ln\left(\dfrac{r_i}{r_{i-1}}\right)}}{2\pi k L}$

will lead to problems at $r_1 = 0$

- alternative rate equation: $R_{LHS} = \dfrac{\text{conduction length}}{\boxed{k\left(\text{conduction area}\right)}} = \dfrac{\Delta r}{k\, 2\pi L\left(r_i - \dfrac{\Delta r}{2}\right)}$

must be evaluated at the interface!
you must get the same value on both sides of the interface

Rate Equations

Internal Nodes:

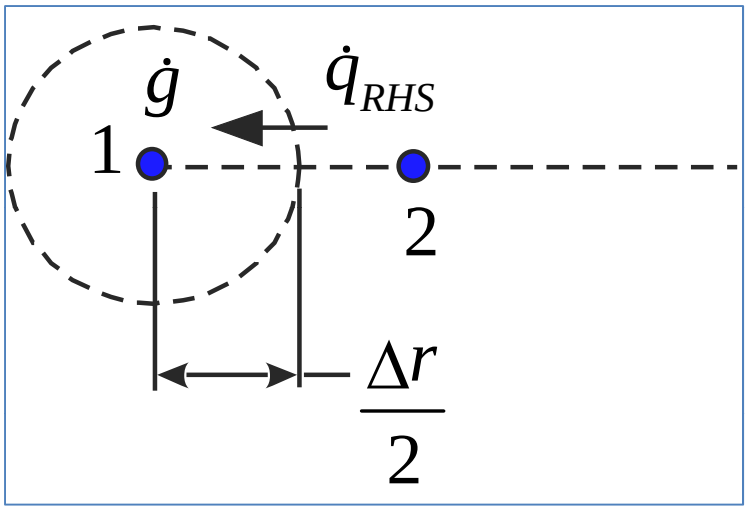
$$\underbrace{\frac{2\pi L k}{\Delta r}\left(r_i - \frac{\Delta r}{2}\right)\left(T_{i-1} - T_i\right)}_{\dot{q}_{LHS}} + \underbrace{\frac{2\pi L k}{\Delta r}\left(r_i + \frac{\Delta r}{2}\right)\left(T_{i+1} - T_i\right)}_{\dot{q}_{RHS}} + \underbrace{\dot{g}'''\, 2\pi r_i L \Delta r}_{\dot{g}} = 0$$

$$\text{for } i = \underbrace{2..(N-1)}_{\text{internal nodes}}$$

- N-2 equations in N unknowns (T_1 ... T_N)
- Remaining two equations from boundary nodes

Boundary Node (@ Center)

Node 1:



Energy balance: $\dot{q}_{RHS} + \dot{g} = 0$

Rate equations:

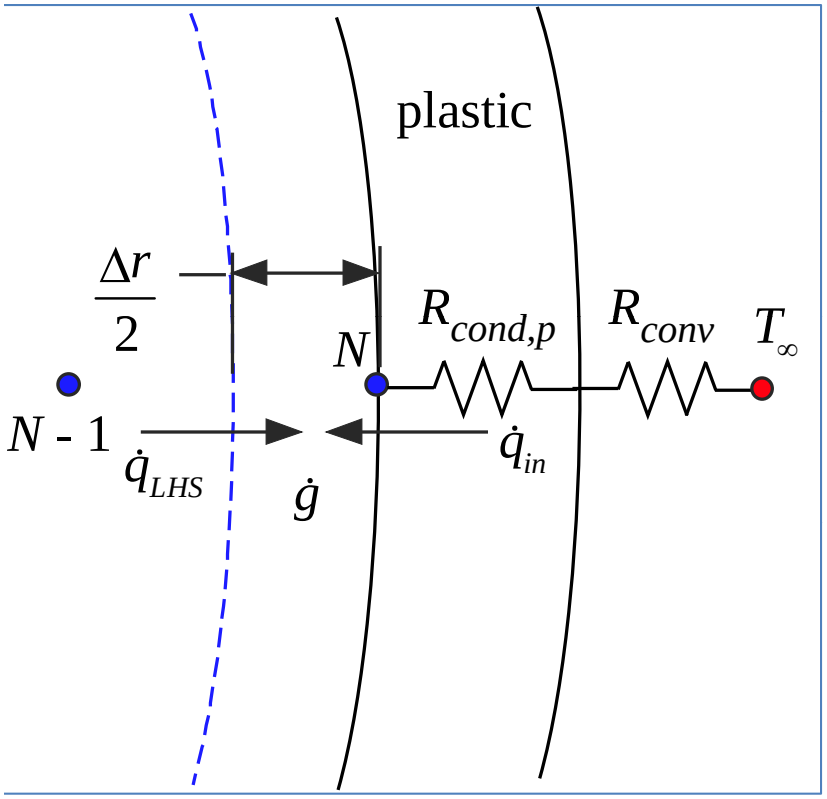
$$\dot{q}_{RHS} = \underbrace{2\pi L\left(\frac{\Delta r}{2}\right)}_{\text{area of interface}} \frac{k}{\underbrace{\Delta r}_{\text{conduction length}}} \underbrace{\left(T_2 - T_1\right)}_{\text{temperature difference}}$$

$$\dot{g} = \underbrace{\pi L\left(\frac{\Delta r}{2}\right)^2}_{\text{volume of CV}} \dot{g}'''$$

$$\boxed{\frac{2\pi L k}{\Delta r}\left(\frac{\Delta r}{2}\right)\left(T_2 - T_1\right) + \pi L\left(\frac{\Delta r}{2}\right)^2 \dot{g}''' = 0}$$

Boundary Node (at r = R)

Node N:



Energy balance: $\dot{q}_{LHS} + \dot{q}_{in} + \dot{g} = 0$

Rate equations:

$$\dot{q}_{LHS} = 2\pi L\left(r_N - \frac{\Delta r}{2}\right)\frac{k}{\Delta r}\left(T_{N-1} - T_N\right)$$

$$\dot{q}_{in} = \frac{\left(T_\infty - T_N\right)}{R_{cond,p} + R_{conv}}$$

$$\dot{g} = 2\pi L r_N \frac{\Delta r}{2}\dot{g}'''$$

$$\boxed{\frac{2\pi L k}{\Delta r}\left(r_N - \frac{\Delta r}{2}\right)\left(T_{N-1} - T_N\right) + \frac{\left(T_\infty - T_N\right)}{R_{cond,p} + R_{conv}} + \pi L r_N \Delta r \dot{g}''' = 0}$$

Set of Equations to be Solved

- Numerical models lead to system of algebraic equations (instead of differential equation):

$$\frac{2\pi L k}{\Delta r}\left(\frac{\Delta r}{2}\right)\left(T_2 - T_1\right) + \pi L\left(\frac{\Delta r}{2}\right)^2 \dot{g}''' = 0$$

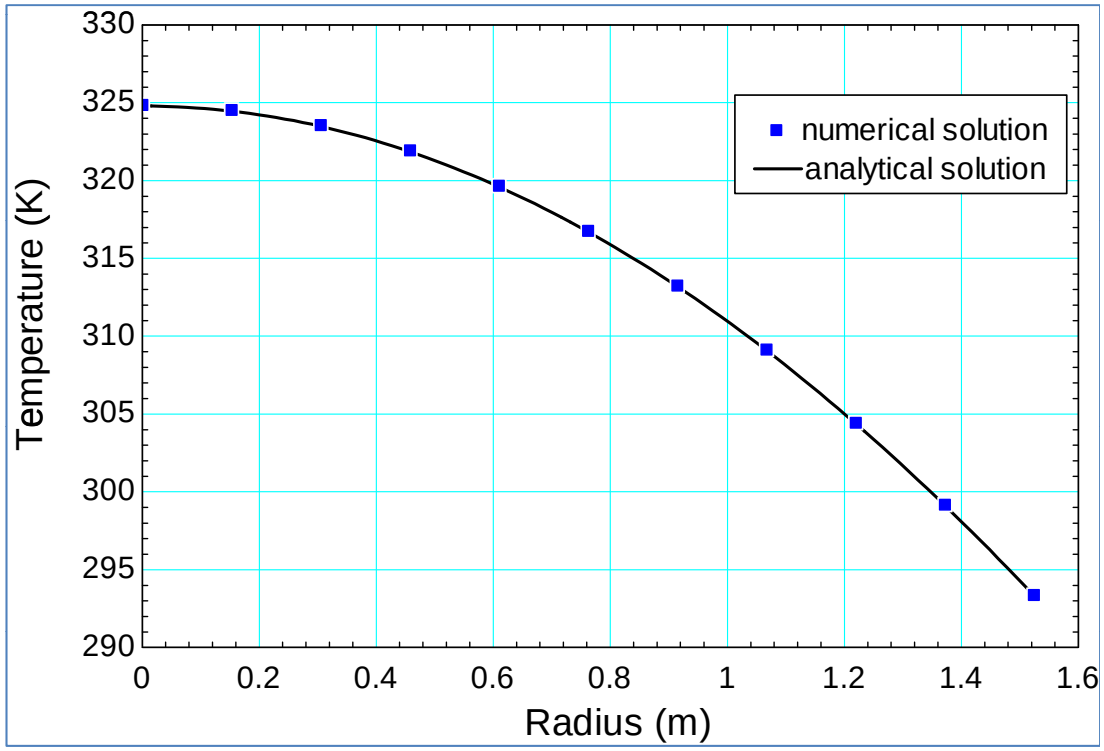
$$\frac{2\pi L k}{\Delta r}\left(r_i - \frac{\Delta r}{2}\right)\left(T_{i-1} - T_i\right) + \frac{2\pi L k}{\Delta r}\left(r_i + \frac{\Delta r}{2}\right)\left(T_{i+1} - T_i\right) + \dot{g}''' 2\pi r_i L \Delta r = 0$$

$$\text{for } i = 2..\left(N-1\right)$$

$$\frac{2\pi L k}{\Delta r}\left(r_N - \frac{\Delta r}{2}\right)\left(T_{N-1} - T_N\right) + \frac{\left(T_\infty - T_N\right)}{R_{cond,p} + R_{conv}} + \pi L r_N \Delta r \dot{g}''' = 0$$

- Solve using appropriate tool, e.g., Matlab

Solution


330
325
320
315
310
305
300
295
290
Temperature (K)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
Radius (m)
numerical solution
analytical solution

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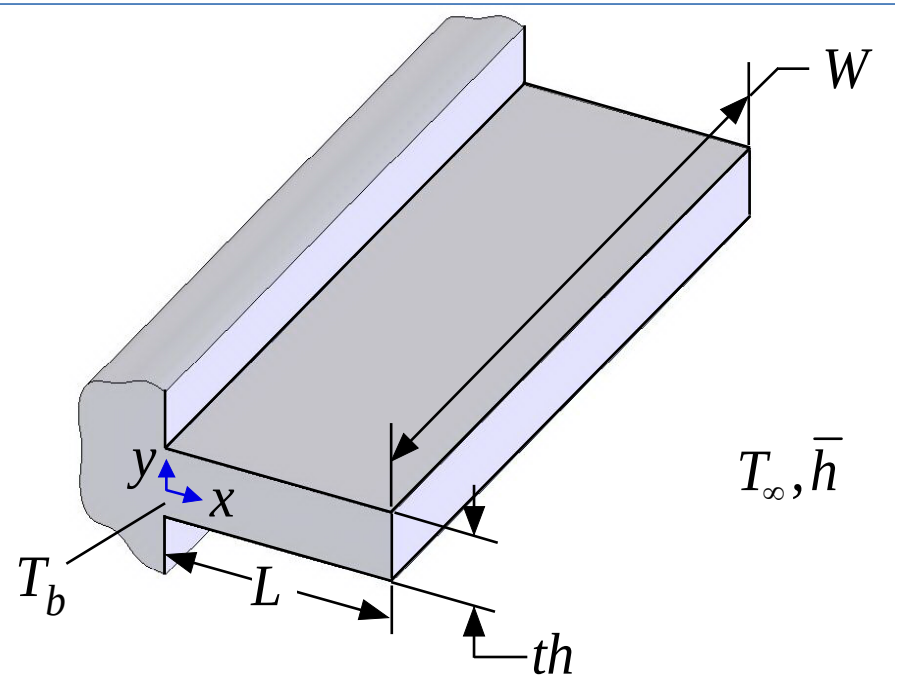
Module 4
Extended Surfaces

1. G. Nellis and S. A. Klein (2009), Heat Transfer, Cambridge University Press Lecture notes provided by Drs. Nellis and Klein

2. Other textbooks such as T.L. Bergman, A.S. Lavine, F.P. Incropera, D.P. DeWitt, "Fundamentals of Heat and Mass Transfer", 8th Edition

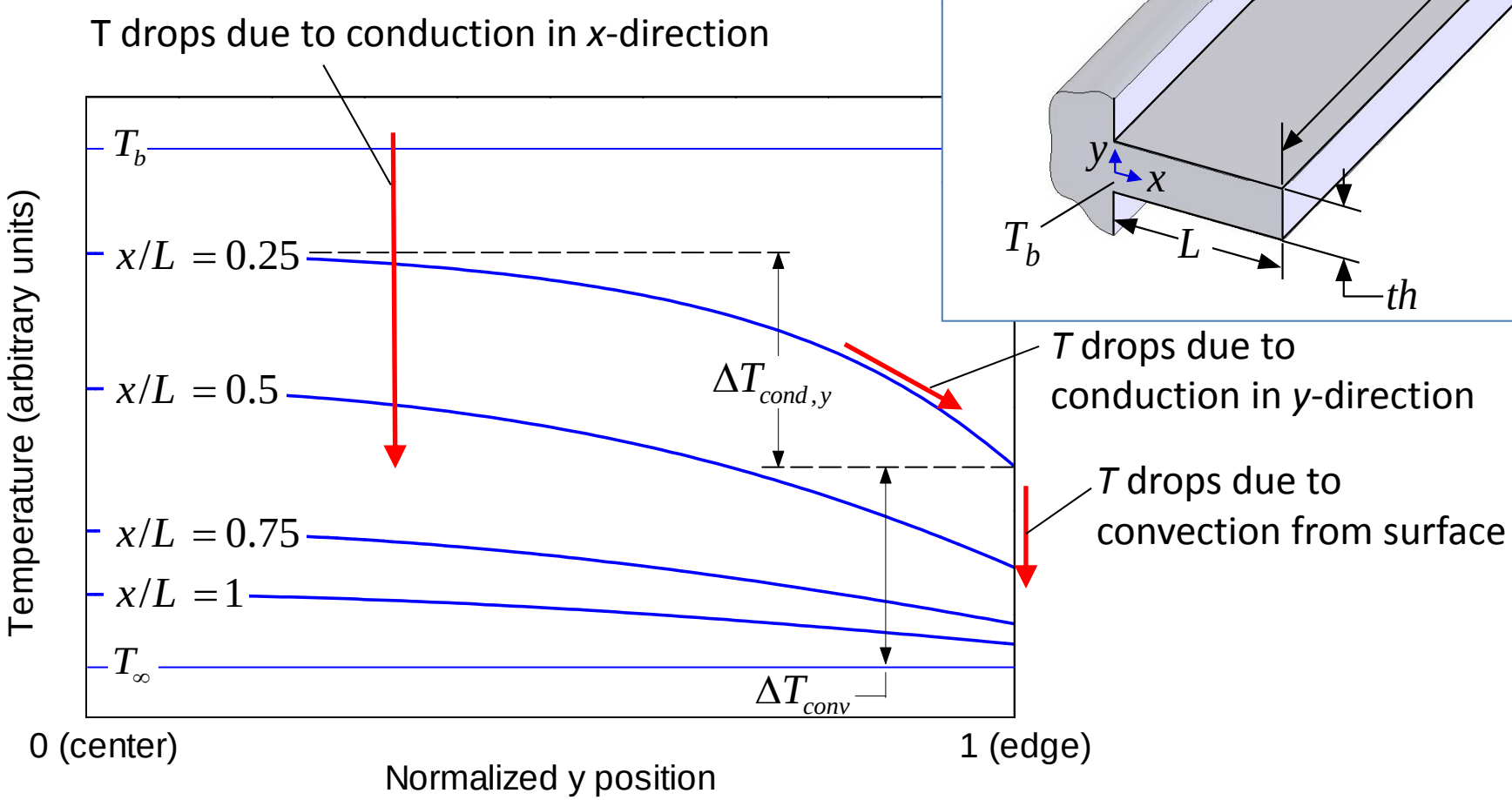
Extended Surface Approximation

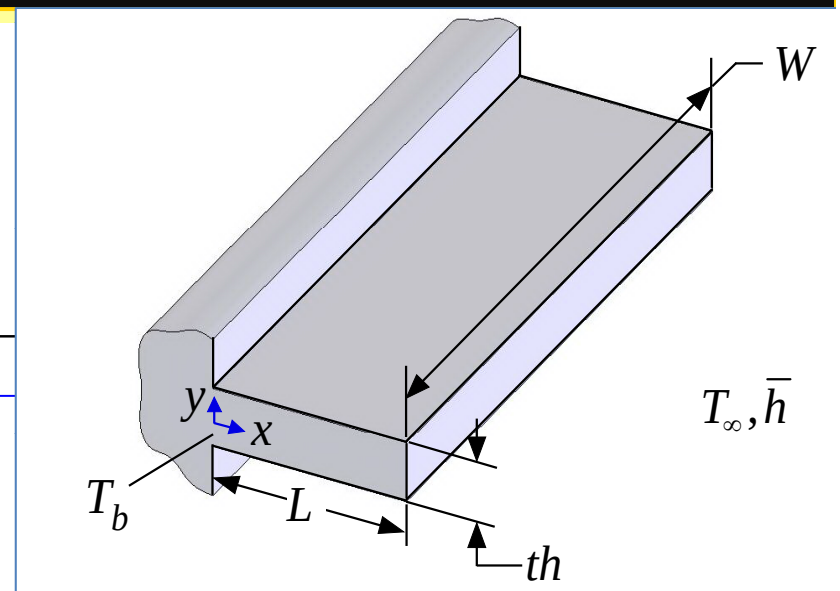
- Many situations where *T* distribution is 2-D/3-D but can approximate as 1-D with little loss of accuracy
- Simplification: extended surface approximation
- The most common situation - fin:
  - thin piece of metal to increase *A* and enhance heat transfer



2-D Temperature Distribution

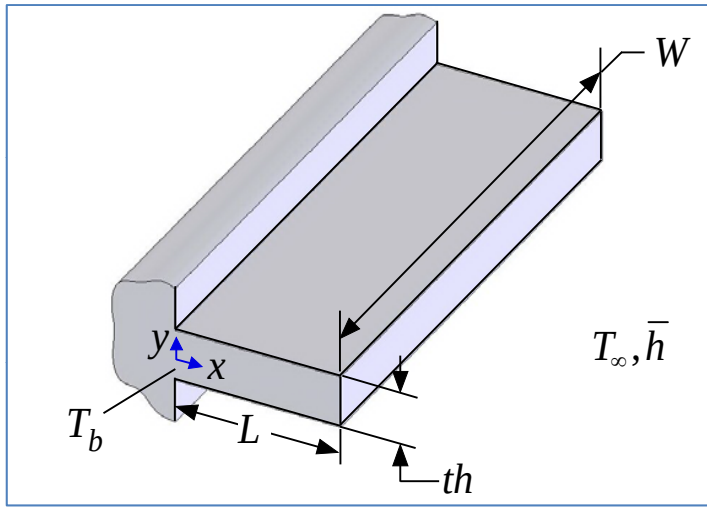
- *T* distribution in fin is 2-D:





Governing Thermal Resistances

- This problem cannot be solved using thermal resistances ("current" is not constant in 1-D)
- However, it can be understood using thermal resistances
- Governing thermal resistances:



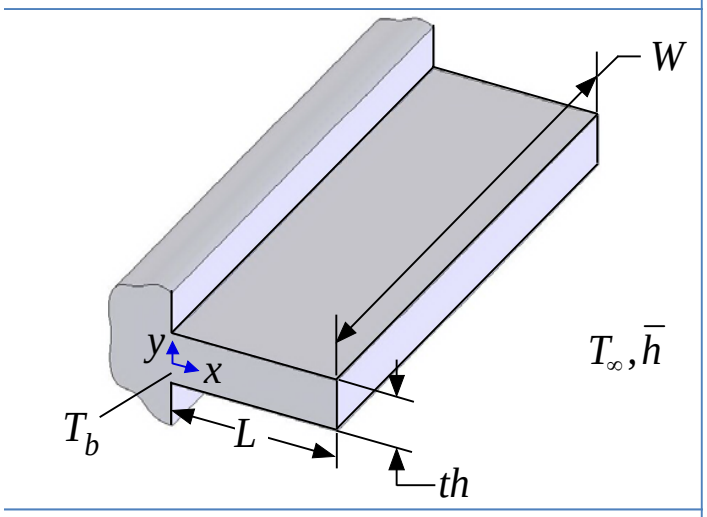
resistance to conduction in x-direction: $R_{cond,x} = \dfrac{L}{kWth}$

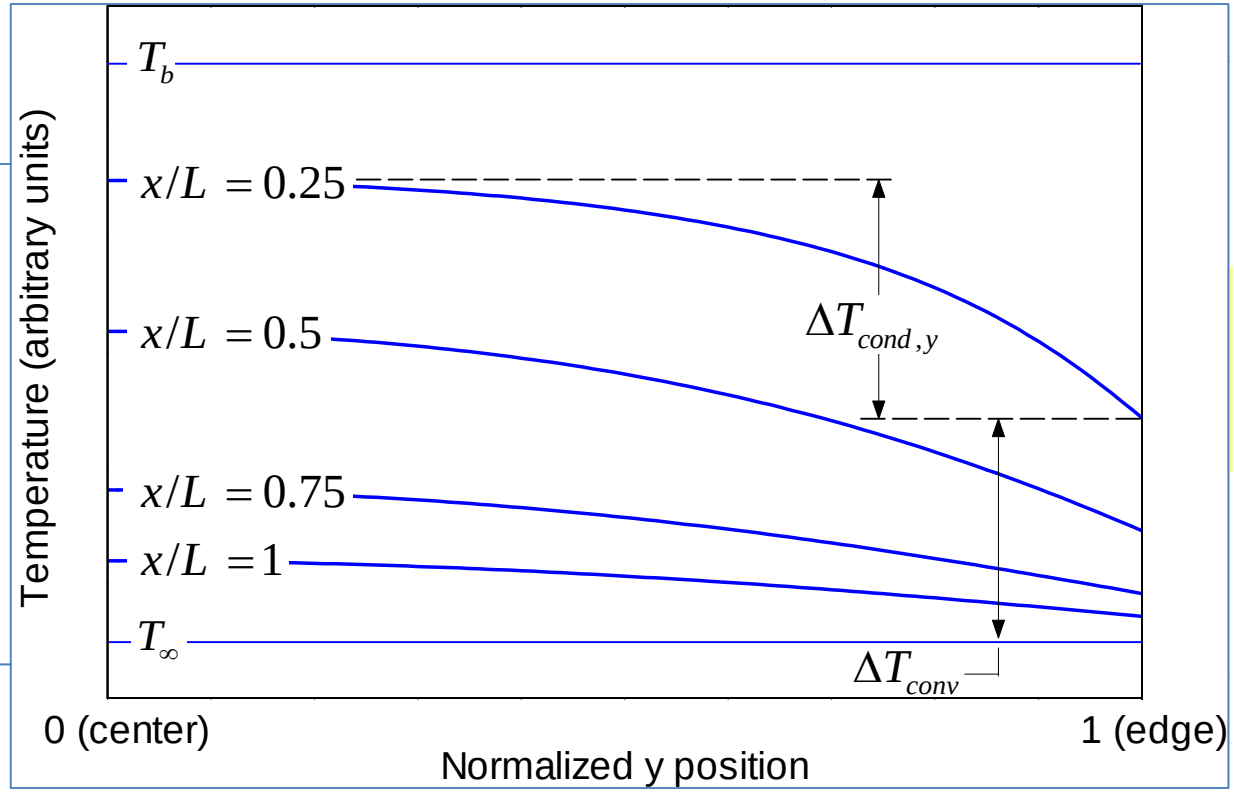
resistance to conduction in y-direction: $R_{cond,y} = \dfrac{th}{4kWL}$

resistance to convection from surface: $R_{conv} = \dfrac{1}{2\bar{h}WL}$

Governing Thermal Resistances

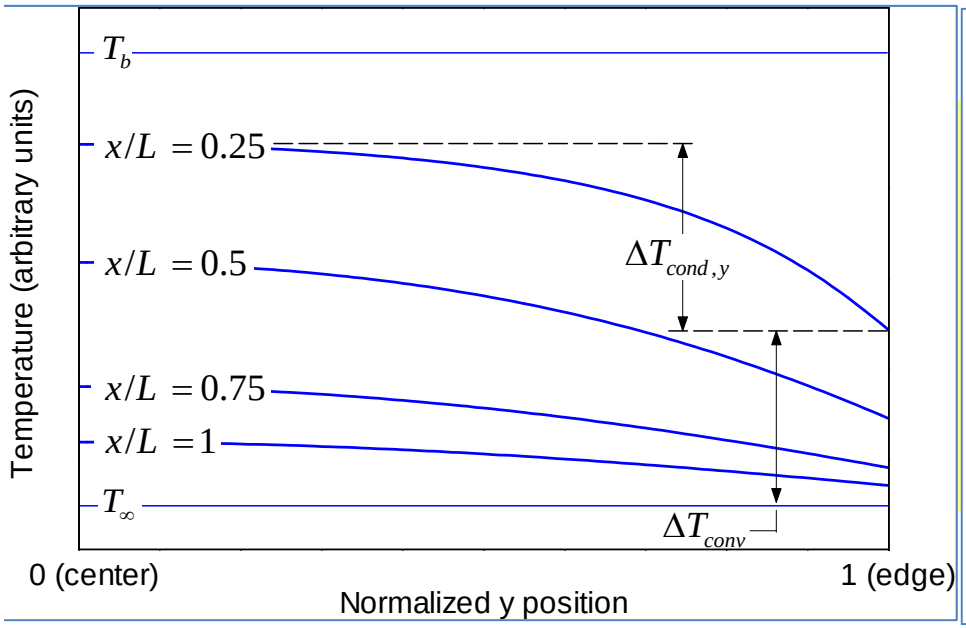
- Understand solution by comparing magnitudes of resistances
- ΔT proportional to resistance

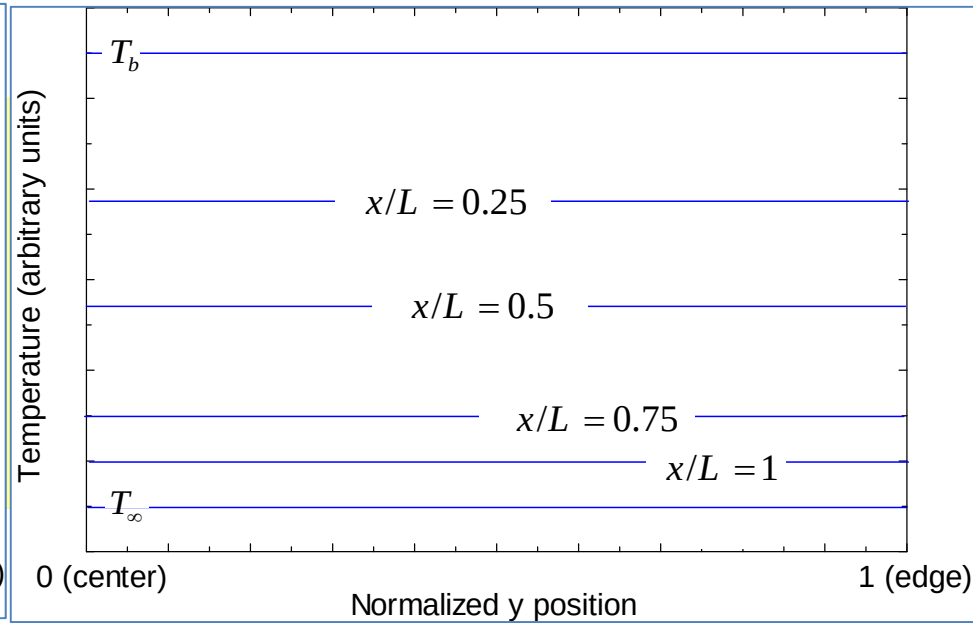




Extended Surface Approximation

- If $R_{cond,y}/R_{conv} << 1$?
- $\Delta T_{cond,y} / \Delta T_{conv} << 1 \Rightarrow$ extended surface approximation
- $\Delta T_{conduction}$ from center of fin to surface small relative to ΔT from the surface to surroundings
- $T(x,y) \sim T(x)$





Biot Number

- Extended surface approximation justified by Biot number

$$\text{Biot number: } Bi = \frac{R_{cond,y}}{R_{conv}} = \frac{th}{4kWL}\frac{2\bar{h}WL}{1} = \frac{th\bar{h}}{2k}$$

- If Bi << 1, extended surface approximation valid
- T~T(x)
- Thin, conductive member with small *h*
- Appropriate Bi not always the same, in general:

$$Bi = \frac{\text{resistance to conduction in direction to "remove"}}{\text{resistance from surface}}$$

$$Bi = \frac{\text{resistance to neglect in your model}}{\text{resistance to consider in your model}}$$

Biot Number for Radiating Fin

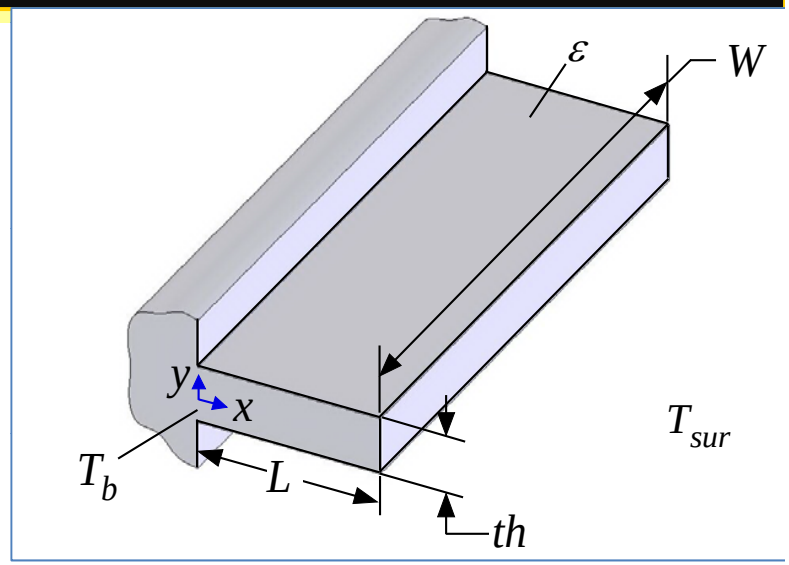
$$Bi = \frac{\text{resistance to conduction in direction to "remove"}}{\text{resistance from surface}}$$

$$Bi = \frac{R_{cond,y}}{R_{rad}}$$

resistance to conduction in y-direction: $R_{cond,y} = \dfrac{th}{4kWL}$

resistance to radiation from the surface: $R_{rad} \approx \dfrac{1}{2\varepsilon WL\sigma\overline{T}^3}$

$$Bi = \frac{th}{4kWL}\frac{2\varepsilon WL\sigma\overline{T}^3}{1} = \frac{th\,\varepsilon\sigma\overline{T}^3}{2k}$$

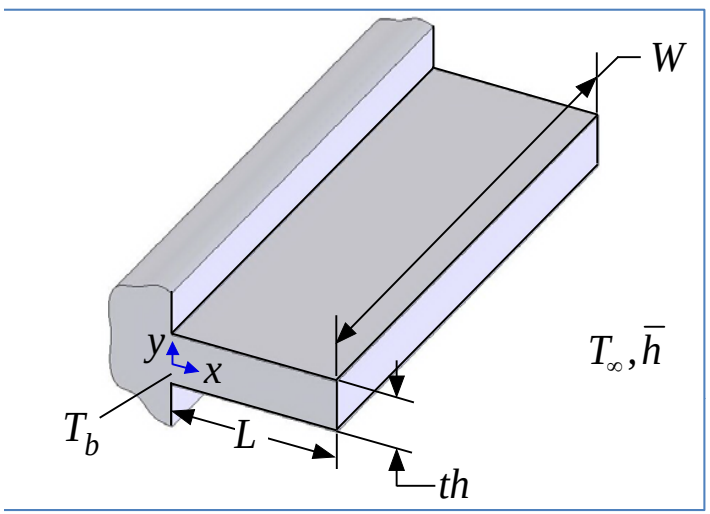


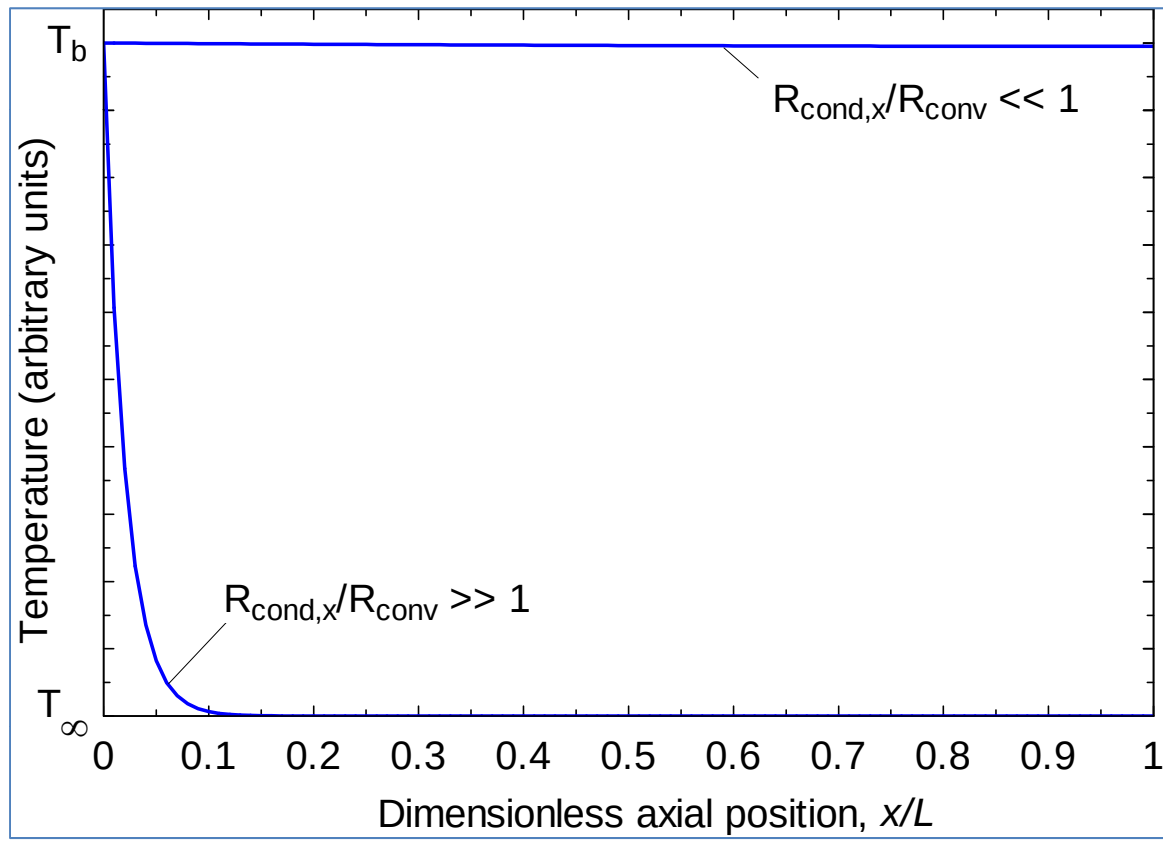
Relative "Resistances"

- If extended surface approximation valid, can anticipate solution based on relative values of R_{conv}, $R_{cond,x}$

$$\frac{R_{cond,x}}{R_{conv}} << 1, \text{ all } \Delta T \text{ due to convection}$$

$$\frac{R_{cond,x}}{R_{conv}} >> 1, \text{ all } \Delta T \text{ due to axial conduction}$$





Governing Equation

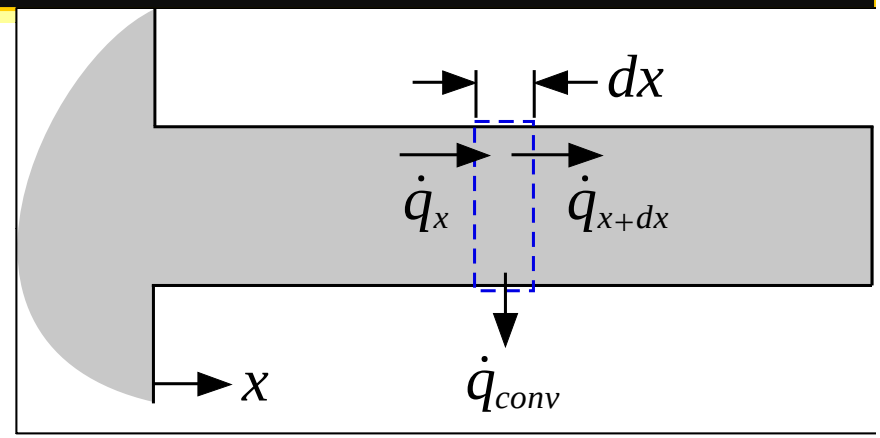
$$\dot{q}_x = \dot{q}_{conv} + \dot{q}_{x+dx}$$

$$\cancel{\dot{q}_x} = \dot{q}_{conv} + \cancel{\dot{q}_x} + \frac{d\dot{q}}{dx}dx$$

$$0 = \dot{q}_{conv} + \frac{d\dot{q}}{dx}dx$$

$$0 = \underbrace{per\,dx\,\bar{h}\left(T - T_\infty\right)}_{\dot{q}_{conv}} + \frac{d}{dx}\underbrace{\left[-k\,A_c\,\frac{dT}{dx}\right]}_{\dot{q}}dx$$

$$\boxed{\frac{d^2T}{dx^2} - \frac{per\,\bar{h}}{k\,A_c}T = -\frac{per\,\bar{h}}{k\,A_c}T_\infty}$$



- Rate of conduction drops due to convection
- Above development is for constant cross section
- Second-order, linear, non-homogeneous (note, if T is a solution then CT is not) ODE
- Not separable
- Split solution into homogeneous (T_h) and particular (T_p) solution: $T = T_h + T_p$

Solution

$$\frac{d^2\left(T_h+T_p\right)}{dx^2}-\frac{per\,\bar{h}}{k\,A_c}\left(T_h+T_p\right)=-\frac{per\,\bar{h}}{k\,A_c}T_\infty$$

$$\underbrace{\frac{d^2T_h}{dx^2}-\frac{per\,\bar{h}}{k\,A_c}T_h}_{\substack{=0\\ \text{for homogeneous}\\ \text{differential equation}}}+\underbrace{\frac{d^2T_p}{dx^2}-\frac{per\,\bar{h}}{k\,A_c}T_p=-\frac{per\,\bar{h}}{k\,A_c}T_\infty}_{\substack{\text{whatever is left over must be the}\\ \text{particular differential equation}}}$$

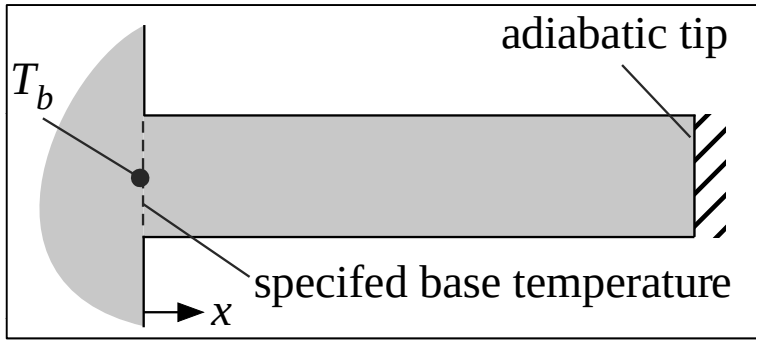
- Particular ODE: by inspection, T_p = constant should work. Substitute into particular ODE to get $\boxed{T_p=T_\infty}$
- Homogeneous ODE: $T_h=C\exp\left(m\,x\right)$

$$C\,m^2\exp\left(m\,x\right)-\frac{per\,\bar{h}}{k\,A_c}C\exp\left(m\,x\right)=0;\quad m^2=\frac{per\,\bar{h}}{k\,A_c}$$

- Two solutions corrseponding to two roots of quadratic equation $T_{h,1}=C_1\exp\left(m\,x\right);\quad T_{h,2}=C_2\exp\left(-m\,x\right)$

Solution

- General solution: $T = T_h + T_p = C_1 \exp(mx) + C_2 \exp(-mx) + T_\infty$
- Constants based on BC
- Base: $\boxed{T_{x=0} = T_b}$
- Fin tip: several possibilities
- Consider adiabatic tip $\dot{q}_{x=L} = 0;$ $\boxed{-kA_c \left(\frac{dT}{dx}\right)_{x=L} = 0}$
- Substitute into general solution to get:



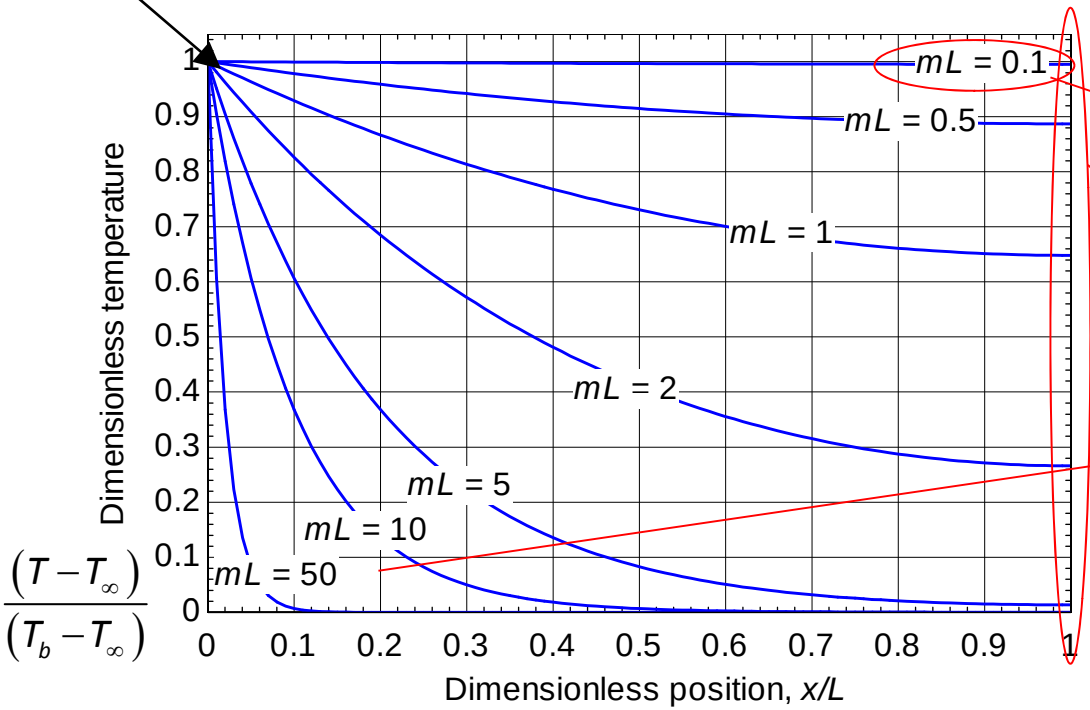
$$T = \underbrace{\frac{(T_b - T_\infty)\exp(-mL)}{\exp(-mL) + \exp(mL)}}_{C_1} \exp(mx) + \underbrace{\frac{(T_b - T_\infty)\exp(mL)}{\exp(-mL) + \exp(mL)}}_{C_2} \exp(-mx) + T_\infty$$

$$\boxed{T = (T_b - T_\infty)\frac{\left[\exp(-m(L-x)) + \exp(m(L-x))\right]}{\left[\exp(-mL) + \exp(mL)\right]} + T_\infty}; \quad \textit{alternatively} \quad \boxed{\frac{(T - T_\infty)}{(T_b - T_\infty)} = \frac{\cosh\left[mL\left(1 - \frac{x}{L}\right)\right]}{\cosh(mL)}}$$

Interpretation

$$R_{cond,x} = \frac{L}{k A_c} \qquad R_{conv} = \frac{1}{\bar{h}\, per\, L} \qquad \sqrt{\frac{R_{cond,x}}{R_{conv}}} = \sqrt{\frac{L}{k A_c}\frac{\bar{h}\, per\, L}} = \underbrace{\sqrt{\frac{\bar{h}\, per}{k A_c}}}_{m} L = m L$$

$$T_{x=0} = T_b$$

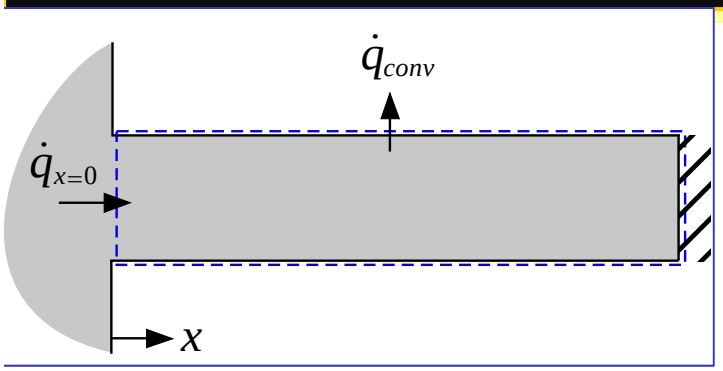


$$\frac{R_{cond,x}}{R_{conv}} \to 0 \text{ fin is isothermal at } T_b$$

$$\left(\frac{dT}{dx}\right)_{x=L} = 0$$

$$\frac{R_{cond,x}}{R_{conv}} \to \infty \text{ fin is isothermal at } T_\infty$$

Fin Heat Transfer

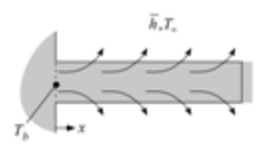
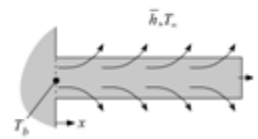
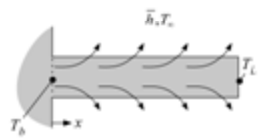
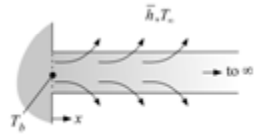


$$\dot{q}_{conv} = \int_0^L \bar{h}\, per\,(T - T_\infty)\,dx$$

$$\dot{q}_{x=0} = -k\,A_c \left(\frac{dT}{dx}\right)_{x=0}$$

$$\dot{q}_{fin} = \sqrt{\bar{h}\, per\, k\, A_c}\,(T_b - T_\infty)\tanh(mL)$$

Other BCs:

End Condition		Temperature distribution
Adiabatic tip		$\frac{T-T_\infty}{T_b-T_\infty} = \frac{\cosh(m(L-x))}{\cosh(mL)}$
		$\dot{q}_{fin} = (T_b - T_\infty)\sqrt{\bar{h}\, per\, k\, A_c}\,\tanh(mL)$
		$\eta_{fin} = \tanh(mL)/(mL)$
Convection from tip		$\frac{T-T_\infty}{T_b-T_\infty} = \frac{\cosh(m(L-x)) + \frac{\bar{h}}{mk}\sinh(m(L-x))}{\cosh(mL) + \frac{\bar{h}}{mk}\sinh(mL)}$
		$\dot{q}_{fin} = (T_b - T_\infty)\sqrt{\bar{h}\, per\, k\, A_c}\,\frac{\sinh(mL) + \frac{\bar{h}}{mk}\cosh(mL)}{\cosh(mL) + \frac{\bar{h}}{mk}\sinh(mL)}$
		$\eta_{fin} = \frac{[\tanh(mL) + mL\,AR_{tip}]}{mL[1 + mL\,AR_{tip}\tanh(mL)](1 + AR_{tip})}$
Specified tip temperature		$\frac{T-T_\infty}{T_b-T_\infty} = \frac{\left[\frac{T_L-T_\infty}{T_b-T_\infty}\right]\sinh(mx) + \sinh(m(L-x))}{\sinh(mL)}$
		$\dot{q}_{fin} = (T_b - T_\infty)\sqrt{\bar{h}\, per\, k\, A_c}\,\frac{\left(\cosh(mL) - \left[\frac{T_L-T_\infty}{T_b-T_\infty}\right]\right)}{\sinh(mL)}$
Infinitely long		$\frac{T-T_\infty}{T_b-T_\infty} = \exp(-mx)$
		$\dot{q}_{fin} = (T_b - T_\infty)\sqrt{\bar{h}\, per\, k\, A_c}$

Fin Efficiency

$$\text{fin efficiency} = \frac{\text{fin heat transfer}}{\text{heat transfer to a "perfect" fin } (k \to \infty)}$$

- Ideal limit corresponds to $R_{\text{cond,x}} = 0$ (fin isothermal @ T_B)

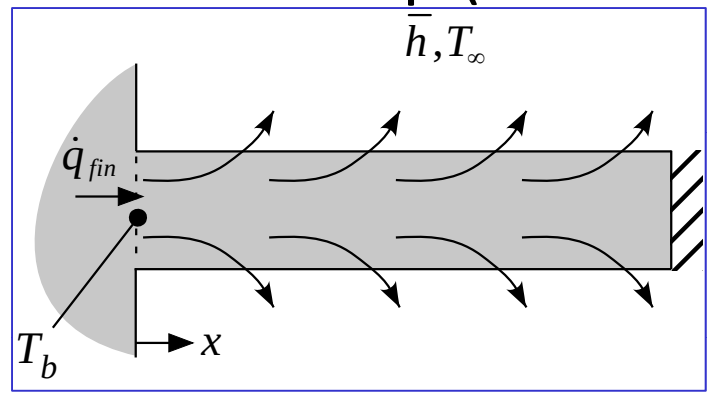
$$\boxed{\eta_{fin} = \frac{\dot{q}_{fin}}{\bar{h} \underbrace{A_{s,fin}}_{\substack{\text{surface area of} \\ \text{fin exposed to fluid}}} (T_b - T_\infty)}}$$

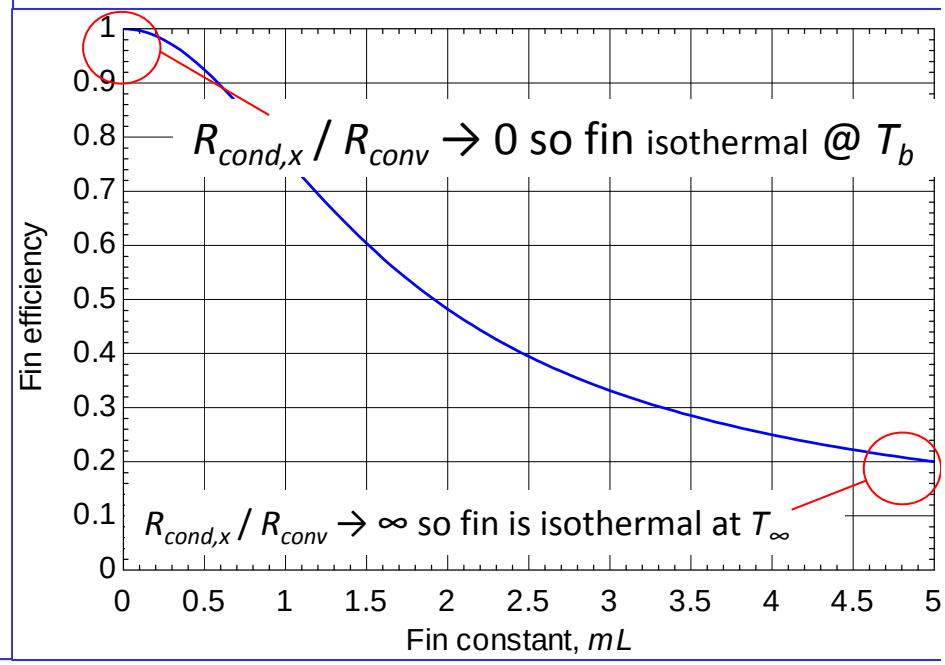
- Definition same for any fin geometry
- For adiabatic tip (const. A_c):

$$\eta_{fin} = \frac{\overbrace{\sqrt{\bar{h}\, per\, k\, A_c}\,(T_b - T_\infty)\tanh(mL)}^{\dot{q}_{fin}}}{\bar{h}\, \underbrace{per\, L}_{A_{s,fin}} (T_b - T_\infty)}$$

$$\eta_{fin} = \frac{\tanh(mL)}{mL}$$

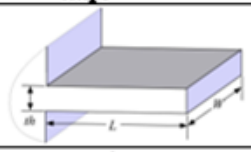
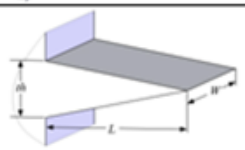
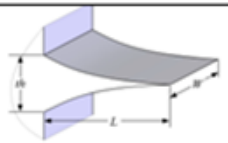
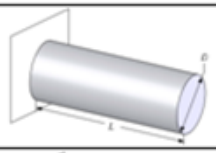
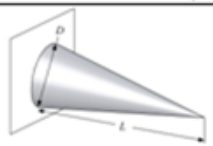
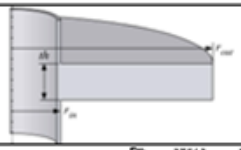
For adiabatic tip (const. A_c):





Fin Efficiencies and Resistances

Table 1-5: Solutions for extended surfaces.

Shape		Efficiency
Straight rectangular		$\eta_{fin} = \frac{\tanh(mL)}{mL}$ $A_{s,fin} = 2WL$ $mL = \sqrt{\frac{2h}{kth}}\,L$
Straight triangular		$\eta_{fin} = \frac{\text{BesselI}(1,2mL)}{mL\,\text{BesselI}(0,2mL)}$ $A_{s,fin} = 2W\sqrt{L^2 + \left(\frac{th}{2}\right)^2}$ $mL = \sqrt{\frac{2h}{kth}}\,L$
Straight parabolic		$\eta_{fin} = \frac{2}{\left[\sqrt{4(mL)^2+1}+1\right]}$ $A_{s,fin} = W\left[C_1 L + \frac{L}{th}\ln\left(\frac{th}{L}+C_1\right)\right]$ $mL = \sqrt{\frac{2h}{kth}}\,L,\ C_1 = \sqrt{1+\left(\frac{th}{L}\right)^2}$
Spine rectangular		$\eta_{fin} = \frac{\tanh(mL)}{mL}$ $A_{s,fin} = \pi DL$ $mL = \sqrt{\frac{4h}{kD}}\,L$
Spine triangular		$\eta_{fin} = \frac{2\,\text{BesselI}(2,2mL)}{mL\,\text{BesselI}(1,2mL)}$ $A_{s,fin} = \frac{\pi D}{2}\sqrt{L^2 + \left(\frac{D}{2}\right)^2}$ $mL = \sqrt{\frac{4h}{kD}}\,L$
Rectangular annular		$A_{s,fin} = 2\pi(r_{2c}^2 - r_1^2)$ $mr_1 = \sqrt{\frac{2h}{kth}}\,r_1$ $mr_{2c} = \sqrt{\frac{2h}{kth}}\,r_{2c}$
	$\eta_{fin} = \frac{2mr_{2c}}{\left[(mr_{2c})^2 - (mr_1)^2\right]}\frac{\left[\text{BesselK}(1,mr_1)\text{BesselI}(1,mr_{2c}) - \text{BesselI}(1,mr_1)\text{BesselK}(1,mr_{2c})\right]}{\left[\text{BesselI}(0,mr_1)\text{BesselK}(1,mr_{2c}) + \text{BesselK}(0,mr_1)\text{BesselI}(1,mr_{2c})\right]}$	
where	h = heat transfer coefficient	k = thermal conductivity

- R_{fin} convenient method of incorporating fin solutions into larger engineering problem

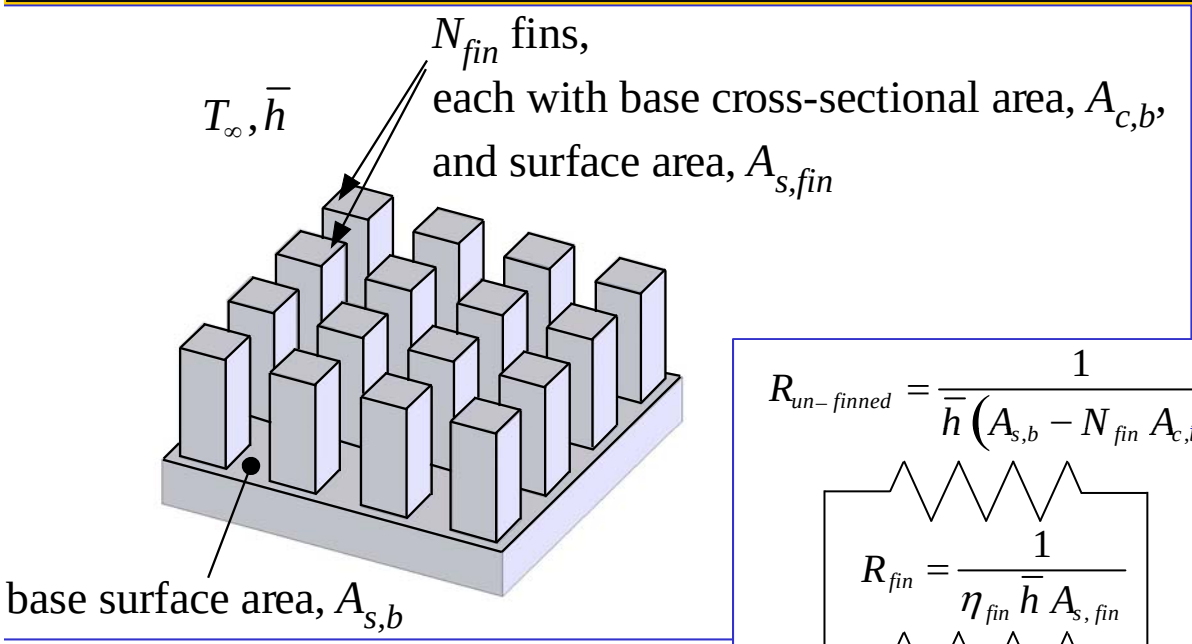
$$\eta_{fin} = \frac{\dot{q}_{fin}}{\bar{h}\,A_{s,fin}\left(T_b - T_\infty\right)} \longrightarrow \dot{q}_{fin} = \underbrace{\eta_{fin}\,\bar{h}\,A_{s,fin}}_{\frac{1}{R_{fin}}}\left(T_b - T_\infty\right)$$

$$\dot{q}_{fin} = \frac{\left(T_b - T_\infty\right)}{R_{fin}} \quad \text{where} \quad \boxed{R_{fin} = \frac{1}{\eta_{fin}\,\bar{h}\,A_{s,fin}}}$$

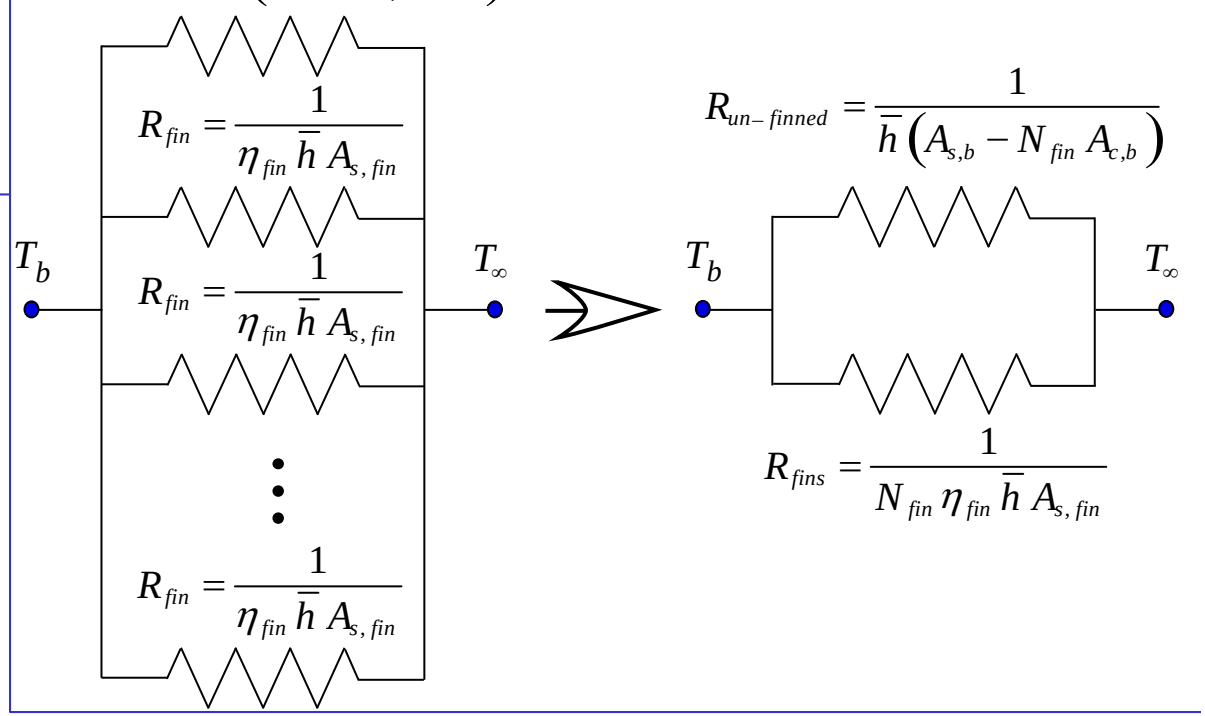
R_{fin} incorporates effect of convection and conduction

- $R_{fin} > R_{conv} = 1/(hA_{s,fin})$ because $\eta_{fin} < 1$
- R_{fin} evaluated using appropriate η_{fin} solution
- R_{fin} can be inserted into resistance network

Finned Surfaces



$$R_{un-finned} = \frac{1}{\bar{h}\left(A_{s,b} - N_{fin} A_{c,b}\right)}$$

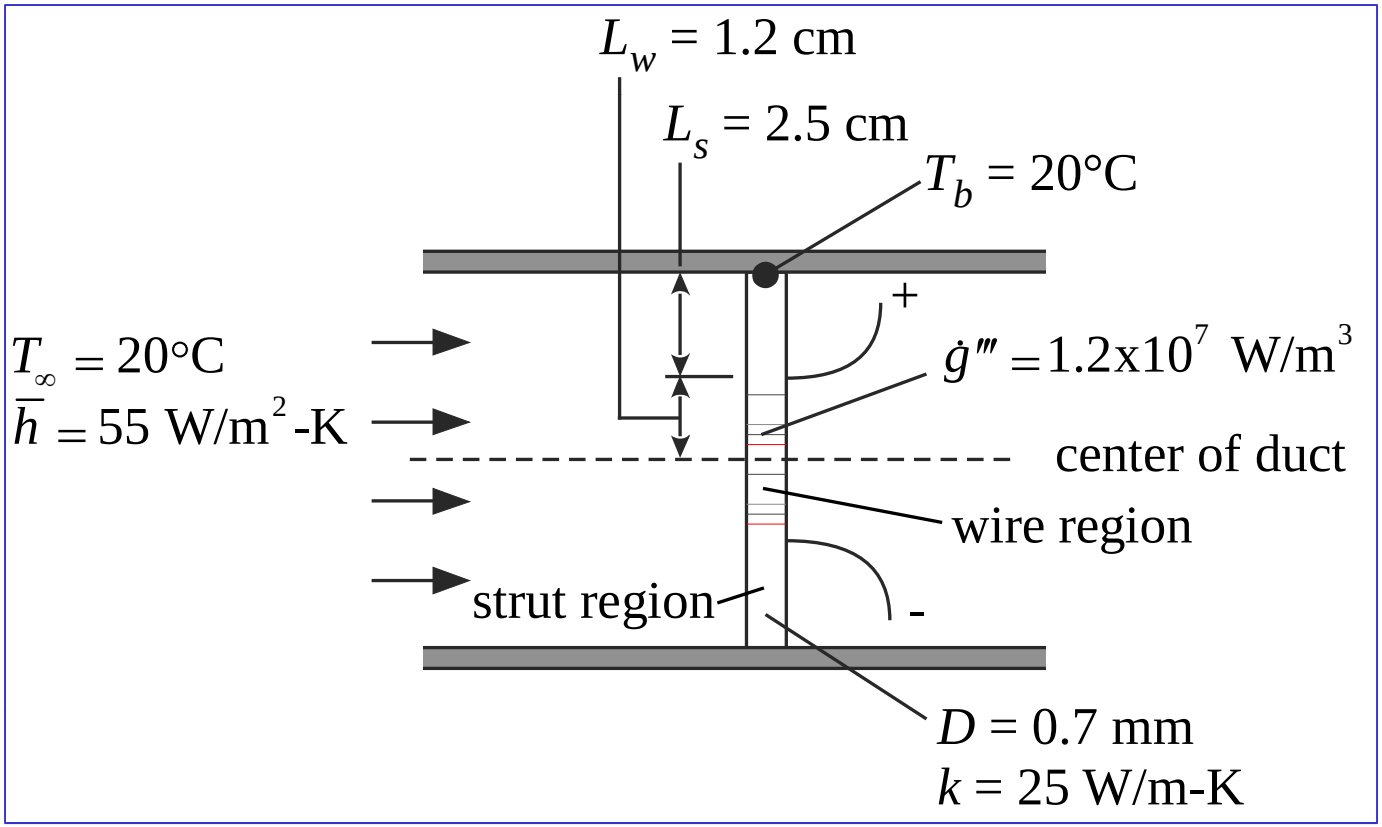


Extended Surface Solutions

- Extended surface approximation can be used to analyze situations other than fins
  - volumetric generation
  - heat flux
  - multiple computational domains

Hot Wire Example

- Hot wire mounted in a duct to measure velocity
- Only the middle portion (the wire region) exposed to volumetric heating to avoid edge effects

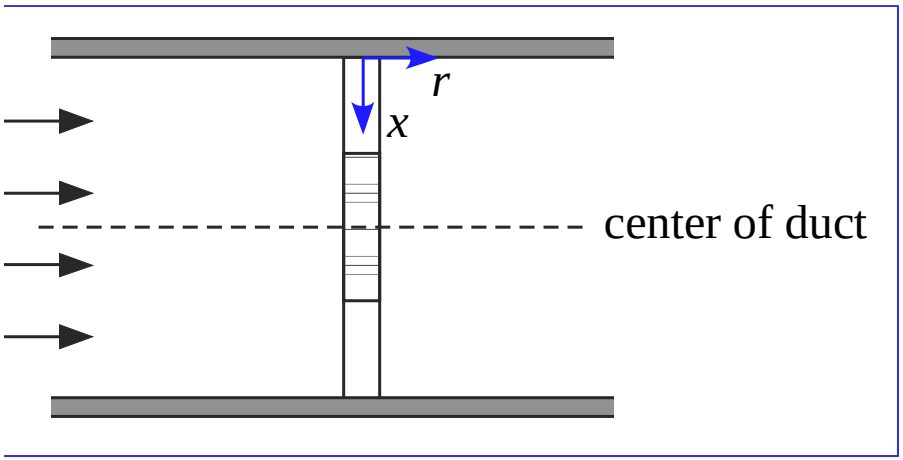


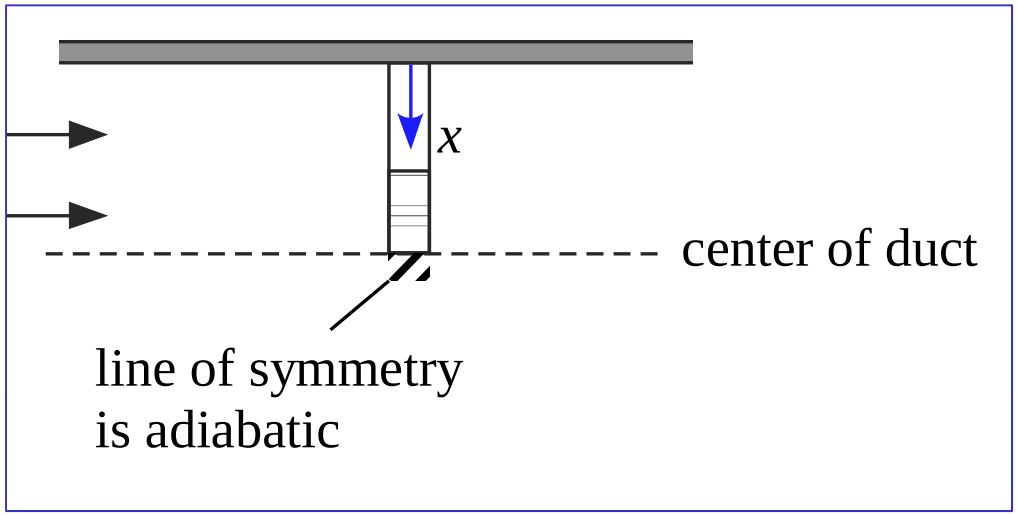
Extended Surface Approximation

- Extended surface approximation allows us to assume $T(x)$ not $T(x,r)$
- Approximation justified with:

$$Bi = \frac{R_{cond,r}}{R_{conv}}; \quad Bi \approx \frac{\overbrace{(D/2)}^{\text{conduction length}}}{k \underbrace{\pi D L}_{\text{conduction area}}} \bar{h} \pi D L; \quad Bi \approx \frac{\bar{h} D}{2k} = 8\text{x}10^{-4}$$

<< 1 so extended surface approximation is appropriate





Two Regions

- Two computational domains:
  - $0 < x < L_s$: no volumetric heating (T_s)
  - $L_s < x < L_s + L_w$: volumetric heating (T_w)

$$\cancel{\dot{q}_x} + \dot{g} = \cancel{\dot{q}_x} + \frac{d\dot{q}}{dx}dx + \dot{q}_{conv}$$

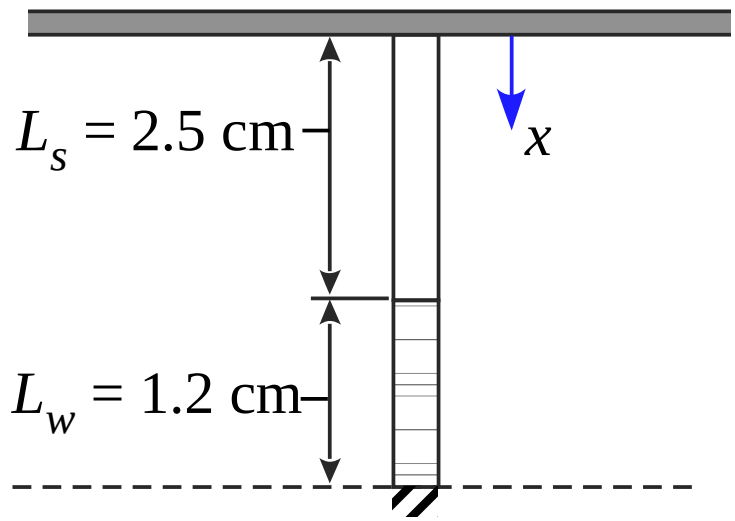
$$\dot{g}''' A_c\, dx = \frac{d}{dx}\left[-k A_c \frac{dT_w}{dx}\right]dx + \bar{h}\, per\, dx\left(T_w - T_\infty\right)$$

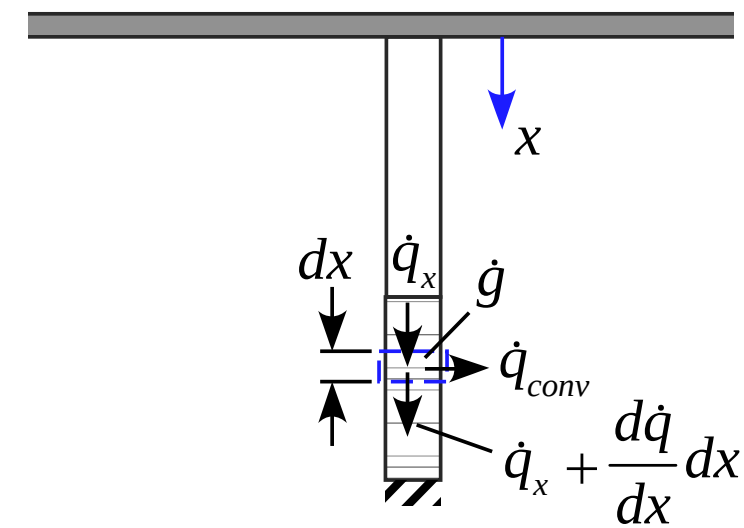
$$\boxed{\frac{d^2T_w}{dx^2} - \frac{\bar{h}\, per}{k A_c}T_w = -\frac{\bar{h}\, per}{k A_c}T_\infty - \frac{\dot{g}'''}{k}}$$

ODE for T_w
$L_s < x < (L_s + L_w)$

$$\boxed{\frac{d^2T_s}{dx^2} - \frac{\bar{h}\, per}{k A_c}T_s = -\frac{\bar{h}\, per}{k A_c}T_\infty}$$

ODE for T_s
$0 < x < L_s$





General Solutions

- Split into particular and homogeneous DE

$$\frac{d^2T_w}{dx^2} - \underbrace{\frac{\bar{h}\,per}{k\,A_c}}_{m^2} T_w = -\frac{\bar{h}\,per}{k\,A_c} T_\infty - \frac{\dot{g}'''}{k}$$

$$\frac{d^2T_{w,h}}{dx^2} - m^2\,T_{w,h} = 0$$

$$\frac{d^2T_{w,p}}{dx^2} - \frac{\bar{h}\,per}{k\,A_c} T_{w,p} = -\frac{\bar{h}\,per}{k\,A_c} T_\infty - \frac{\dot{g}'''}{k}$$

$$T_{w,h} = C_1 \sinh(m\,x) + C_2 \cosh(m\,x)$$

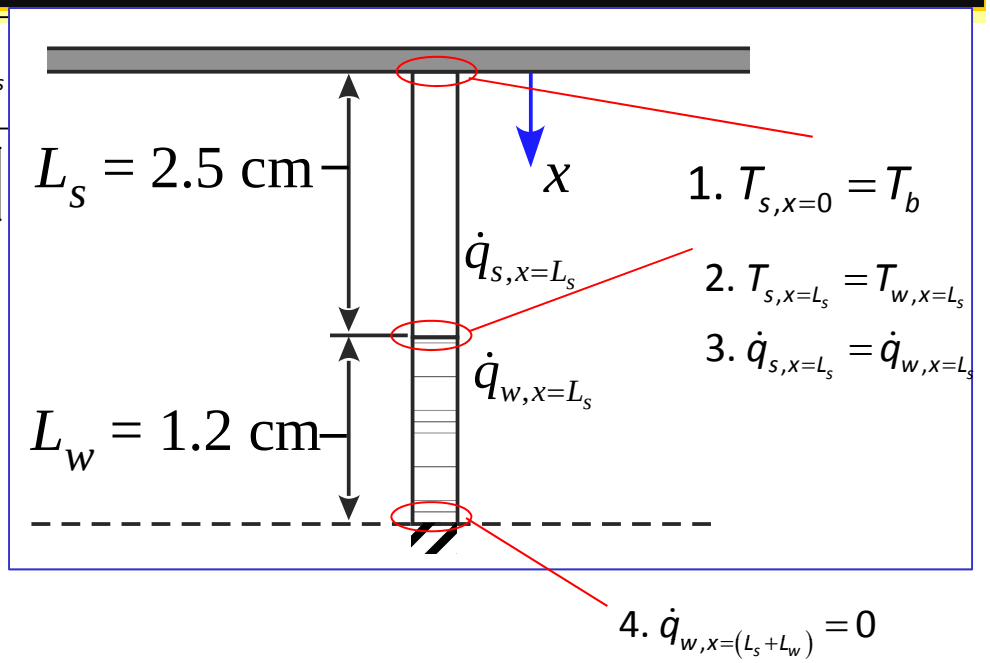
$$T_{w,p} = T_\infty + \frac{\dot{g}'''\,A_c}{\bar{h}\,per}$$

$$T_w = T_{w,h} + T_{w,p} = \boxed{C_1 \sinh(m\,x) + C_2 \cosh(m\,x) + T_\infty + \frac{\dot{g}'''\,A_c}{\bar{h}\,per}}$$

Boundary Conditions

$$T_w = C_1 \sinh(mx) + C_2 \cosh(mx) + T_\infty + \frac{\dot{g}''' A_c}{\bar{h}\, per} \quad \text{for } L_s < x < (L_s$$

$$T_s = C_3 \sinh(mx) + C_4 \cosh(mx) + T_\infty \quad \text{for } 0 < x < L_s$$



$$C_3 \underbrace{\sinh(m0)}_{=0} + C_4 \underbrace{\cosh(m0)}_{=1} + T_\infty = T_b \qquad \boxed{C_4 + T_\infty = T_b}$$

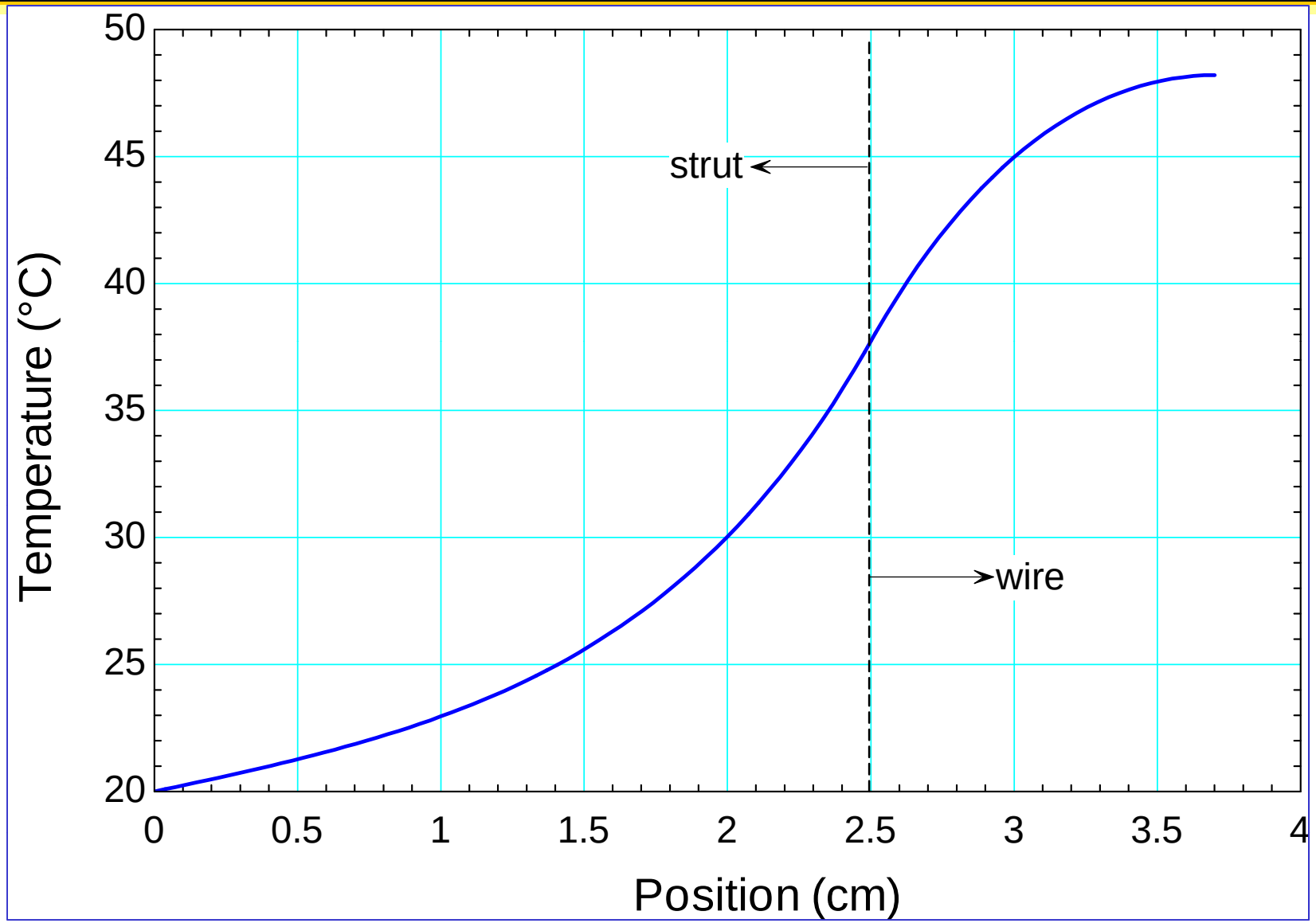
$$C_3 \sinh(mL_s) + C_4 \cosh(mL_s) + T_\infty = C_1 \sinh(mL_s) + C_2 \cosh(mL_s) + T_\infty + \frac{\dot{g}''' A_c}{\bar{h}\, per}$$

$$C_3\, m \sinh(mL_s) + C_4\, m \cosh(mL_s) = C_1\, m \sinh(mL_s) + C_2\, m \cosh(mL_s)$$

$$C_1\, m \sinh(m(L_s + L_w)) + C_2\, m \cosh(m(L_s + L_w)) = 0$$

4 Equations
in 4 unknowns, can solve in EES using the appropriate values for heat generation, h, and *T*s

Temperature Distribution



Non Constant Cross Section Extended Surfaces

- Constant cross-section extended surfaces yield solutions in terms of exponentials and/or hyberbolic functions, which are series solutions with the appropriate constants.
- Homogeneous differential equation that results from analysis of extended surfaces with non-constant cross-section (and many other problems) is:

$$\frac{d}{dx}\left(x^p \frac{d\theta}{dx}\right) \pm c^2 x^s \theta = 0$$

(Bessel's equation)

- Series solutions to this ODE referred to as Bessel functions
- Rules for using Bessel functions are well-defined

Bessel Function Solutions

Governing differential equation:

$$\frac{d}{dx}\left(x^p \frac{d\theta}{dx}\right) \pm c^2 x^s \theta = 0$$

$s - p + 2 \neq 0$

Calculate solution parameters:

$$n = \frac{(1-p)}{(s-p+2)} \quad a = \frac{2}{(s-p+2)} \quad \frac{n}{a} = \frac{(1-p)}{2}$$

Last term in dif. eq. is negative:

$$\frac{d}{dx}\left(x^p \frac{d\theta}{dx}\right) - c^2 x^s \theta = 0$$

$$\theta = C_1 x^{n/a} \text{BesselI}\left(n, c\,a\,x^{1/a}\right) + C_2 x^{n/a} \text{BesselK}\left(n, c\,a\,x^{1/a}\right)$$

Last term in dif. eq. is positive:

$$\frac{d}{dx}\left(x^p \frac{d\theta}{dx}\right) + c^2 x^s \theta = 0$$

$$\theta = C_1 x^{n/a} \text{BesselJ}\left(n, c\,a\,x^{1/a}\right) + C_2 x^{n/a} \text{BesselY}\left(n, c\,a\,x^{1/a}\right)$$

$s - p + 2 = 0$

$(p-1)^2 - 4c^2 > 0$

Calculate solution parameters:

$$r_1 = \frac{(1-p)}{2} + \sqrt{\frac{(p-1)^2}{4} - c^2}$$

$$r_2 = \frac{(1-p)}{2} - \sqrt{\frac{(p-1)^2}{4} - c^2}$$

$$\theta = C_1 x^{r_1} + C_2 x^{r_2}$$

$(p-1)^2 - 4c^2 = 0$

Calculate solution parameter:

$$d = \frac{(1-p)}{d}$$

$$\theta = C_1 x^d + C_2 x^d \ln(x)$$

$(p-1)^2 - 4c^2 < 0$

Calculate solution parameters:

$$d = \frac{(1-p)}{2}$$

$$e = \sqrt{c^2 - \frac{(p-1)^2}{4}}$$

$$\theta = C_1 x^d \cos\left[e \ln(x)\right] + C_2 x^d \sin\left[e \ln(x)\right]$$

typical solutions for heat transfer problems

Bessel Function Solutions

order of Bessel function

$$\theta = C_1 x^{n/a} \boxed{\mathrm{BesselI}\left(n, c\,a\,x^{1/a}\right)} + C_2 x^{n/a} \boxed{\mathrm{BesselK}\left(n, c\,a\,x^{1/a}\right)}$$

modified Bessel function of the 1st kind

modified Bessel function of the 2nd kind

$$\theta = C_1 x^{n/a} \boxed{\mathrm{BesselJ}\left(n, c\,a\,x^{1/a}\right)} + C_2 x^{n/a} \boxed{\mathrm{BesselY}\left(n, c\,a\,x^{1/a}\right)}$$

Bessel function of the 1st kind

Bessel function of the 2nd kind

Bessel functions resemble sinh and cosh (exponentials)
Modified Bessel functions resemble sine and cosine

Derivatives of Bessel Functions

- Rules provided in text:

$$\frac{d}{dx}\left[\text{BesselI}(0,u)\right] = \text{BesselI}(1,u)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\text{BesselK}(0,u)\right] = -\text{BesselK}(1,u)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\text{BesselJ}(0,u)\right] = -\text{BesselJ}(1,u)\frac{du}{dx}$$

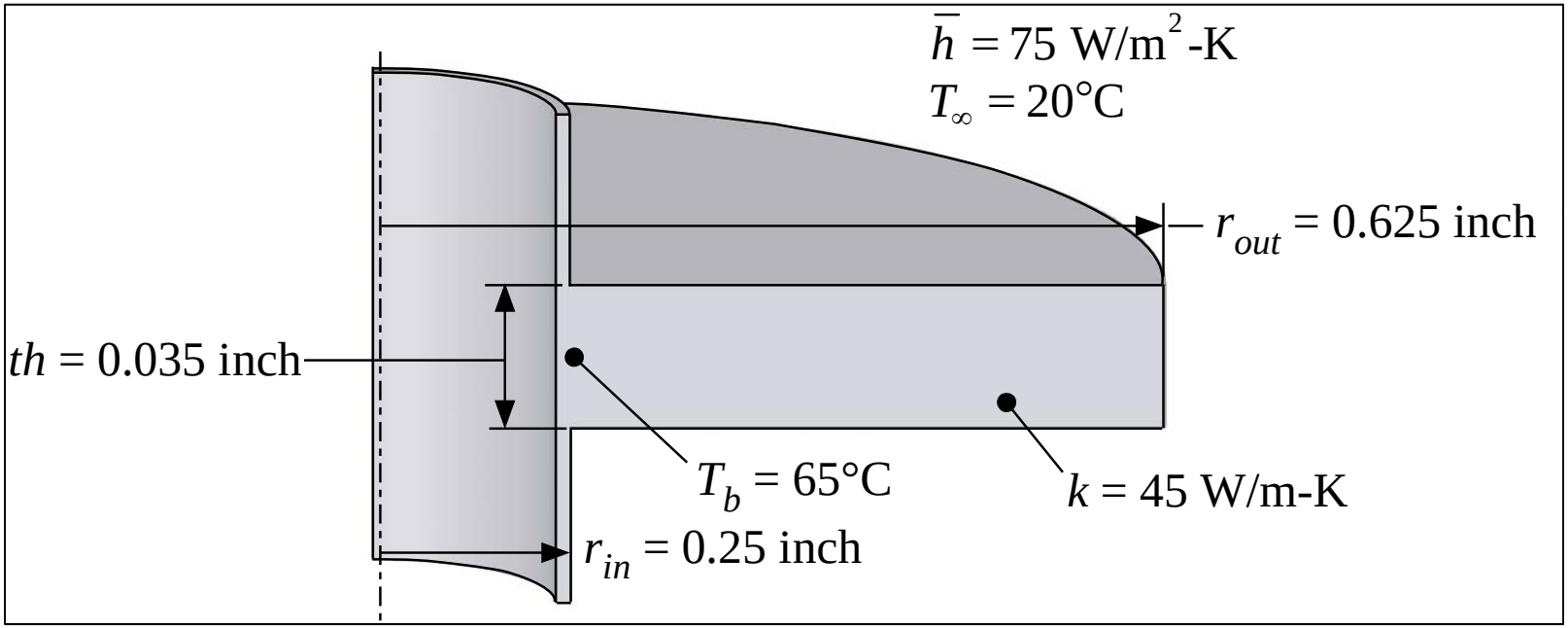
$$\frac{d}{dx}\left[\text{BesselY}(0,u)\right] = -\text{BesselY}(1,u)\frac{du}{dx}$$

etc.

- Maple is also very useful for working with Bessel functions

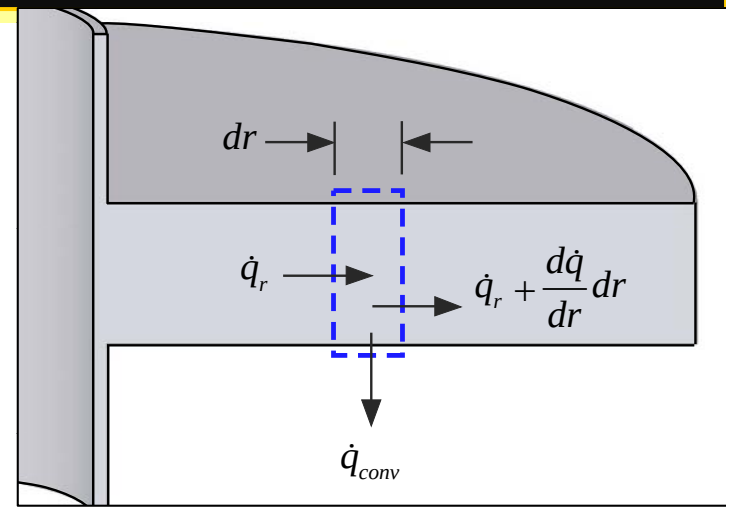
Annular Fin

- Washer-type fin (assume extended surface approximation is valid)



Governing Equation

$$\cancel{\dot{q}_r} = \cancel{\dot{q}_r} + \frac{d\dot{q}}{dr}dr + \dot{q}_{conv}$$



$$\dot{q} = -k\,2\,r\,\pi\,th\frac{dT}{dr}$$

$$\dot{q}_{conv} = 2\,\pi\,r\,dr\,\bar{h}\left(T - T_\infty\right)\boxed{2}$$

convection from two sides

$$\frac{d}{dr}\left(-k\,2\,\boxed{r}\,\pi\,th\frac{dT}{dr}\right)dr + 4\,\pi\,r\,dr\,\bar{h}\left(T - T_\infty\right) = 0$$

you cannot simply remove this from the differential!

$$\boxed{\frac{d}{dr}\left(r\frac{dT}{dr}\right) - \frac{2\bar{h}}{k\,th}r\,T = -\frac{2\bar{h}}{k\,th}r\,T_\infty}$$

$$\boxed{\frac{d}{dr}\left(r\frac{dT}{dr}\right) - m^2\,r\,T = -m^2\,r\,T_\infty \quad \text{where } m = \sqrt{\frac{2\bar{h}}{k\,th}}}$$

particular ODE: $\frac{d}{dr}\left(r\frac{dT_p}{dr}\right) - m^2\,r\,T_p = -m^2\,r\,T_\infty$

homogeneous ODE: $\frac{d}{dr}\left(r\frac{dT_h}{dr}\right) - m^2\,r\,T_h = 0$

Solution

- Particular ODE is solved by inspection:

$$T_p = T_\infty$$

- Homogeneous ODE:

$$\boxed{\frac{d}{dr}\left(r\frac{dT_h}{dr}\right) - m^2\, r\, T_h = 0}$$

- This ODE is a form of Bessel's equation:

$$\frac{d}{dx}\left(x^p \frac{d\theta}{dx}\right) \pm c^2\, x^s\, \theta = 0$$

- Can solve following flow chart with:

$$x = r;\, \theta = T_h;\, p = 1;\, c = m;\, s = 1$$

Flow Chart

$$\frac{d}{dr}\left(r\frac{dT_h}{dr}\right)-m^2\,r\,T_h=0$$

Governing differential equation:

$$\frac{d}{dx}\left(x^p\frac{d\theta}{dx}\right)\pm c^2\,x^s\,\theta=0$$

$s-p+2\neq 0$

$s-p+2=0$

$x=r$

$\theta=T_h$

$p=1$

$c=m$

$s=1$

$$s-p+2=1-1+2=2\neq 0$$

left-hand side of flow chart

Homogeneous Solution

$s - p + 2 \neq 0$

Calculate solution parameters:

$$n = \frac{(1-p)}{(s-p+2)} \quad a = \frac{2}{(s-p+2)} \quad \frac{n}{a} = \frac{(1-p)}{2}$$

$$x = r \qquad \theta = T_h \qquad p = 1 \qquad c = m \qquad s = 1$$

$$n = \frac{1-p}{s-p+2} = \frac{1-1}{1-1+2} = 0$$

$$a = \frac{2}{s-p+2} = \frac{2}{1-1+2} = 1$$

$$\frac{n}{a} = 0$$

Last term in dif. eq. is negative:

$$\frac{d}{dx}\left(x^p \frac{d\theta}{dx}\right) - c^2 x^s \theta = 0$$

$$\theta = C_1 x^{n/a} \text{BesselI}\left(n, c\,a\,x^{1/a}\right) + C_2 x^{n/a} \text{BesselK}\left(n, c\,a\,x^{1/a}\right)$$

last term is negative

$$\theta = C_1 x^{n/a} \text{BesselI}\left(n, c\,a\,x^{1/a}\right) + C_2 x^{n/a} \text{BesselK}\left(n, c\,a\,x^{1/a}\right)$$

$$T_h = C_1 x^0 \text{BesselI}\left(0, m1r^{1/1}\right) + C_2 x^0 \text{BesselK}\left(0, m1r^{1/1}\right)$$

$$\boxed{T_h = C_1 \text{BesselI}(0, m\,r) + C_2 \text{BesselK}(0, m\,r)}$$

Last term in dif. eq. is positive:

$$\frac{d}{dx}\left(x^p \frac{d\theta}{dx}\right) + c^2 x^s \theta = 0$$

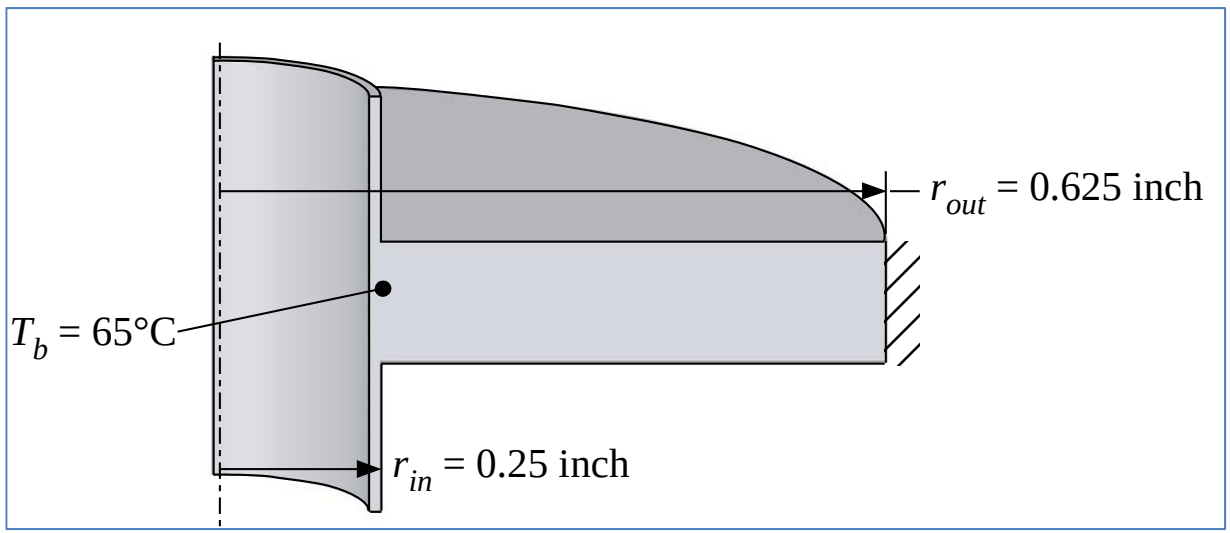
$$\theta = C_1 x^{n/a} \text{BesselJ}\left(n, c\,a\,x^{1/a}\right) + C_2 x^{n/a} \text{BesselY}\left(n, c\,a\,x^{1/a}\right)$$

General Solution

$$T = T_h + T_p$$

$$\boxed{T = C_1 \,\text{BesselI}(0, mr) + C_2 \,\text{BesselK}(0, mr) + T_\infty}$$

- The general solution satisfies the ODE regardless of C1 and C2 - these constants are selected based on the boundary conditions:



$$T_{r=r_{in}} = T_b$$

$$\left(\frac{dT}{dr}\right)_{r=r_{out}} = 0$$

Boundary Conditions

$$T = C_1 \,\text{BesselI}(0, mr) + C_2 \,\text{BesselK}(0, mr) + T_\infty$$

- Substitute the general solution into the BCs

$$T_{r=r_{in}} = T_b \qquad \boxed{T_b = C_1 \,\text{BesselI}(0, mr_{in}) + C_2 \,\text{BesselK}(0, mr_{in}) + T_\infty}$$

$$\left(\frac{dT}{dr}\right)_{r=r_{out}} = 0 \quad C_1 \underbrace{\frac{d}{dr}\left[\text{BesselI}(0, mr)\right]_{r=r_{out}}}_{m\,\text{BesselI}(1, mr_{out})} + C_2 \underbrace{\frac{d}{dr}\left[\text{BesselK}(0, mr)\right]_{r=r_{out}}}_{-m\,\text{BesselK}(1, mr_{out})} = 0$$

$$\boxed{C_1 \, m\,\text{BesselI}(1, mr_{out}) - C_2 \, m\,\text{BesselK}(1, mr_{out}) = 0}$$

- Two equations in two unknowns – solve in EES/Matlab

EES Solution

```
$UnitSystem SI MASS RAD PA  K J
$TABSTOPS   0.2 0.4 0.6 0.8 3.5 in

r_in=0.25 [inch]*convert(inch,m)                "inner radius"
r_out=0.625 [inch]*convert(inch,m)              "outer radius"
th=0.035 [inch]*convert(inch,m)                 "thickness"
h_bar=75 [W/m^2-K]                              "heat transfer coefficient"
T_b=converttemp(C,K,65 [C])                     "base temperature"
T_infinity=converttemp(C,K,20 [C])              "ambient temperature"
k=45 [W/m-K]                                    "conductivity"
```

Inputs

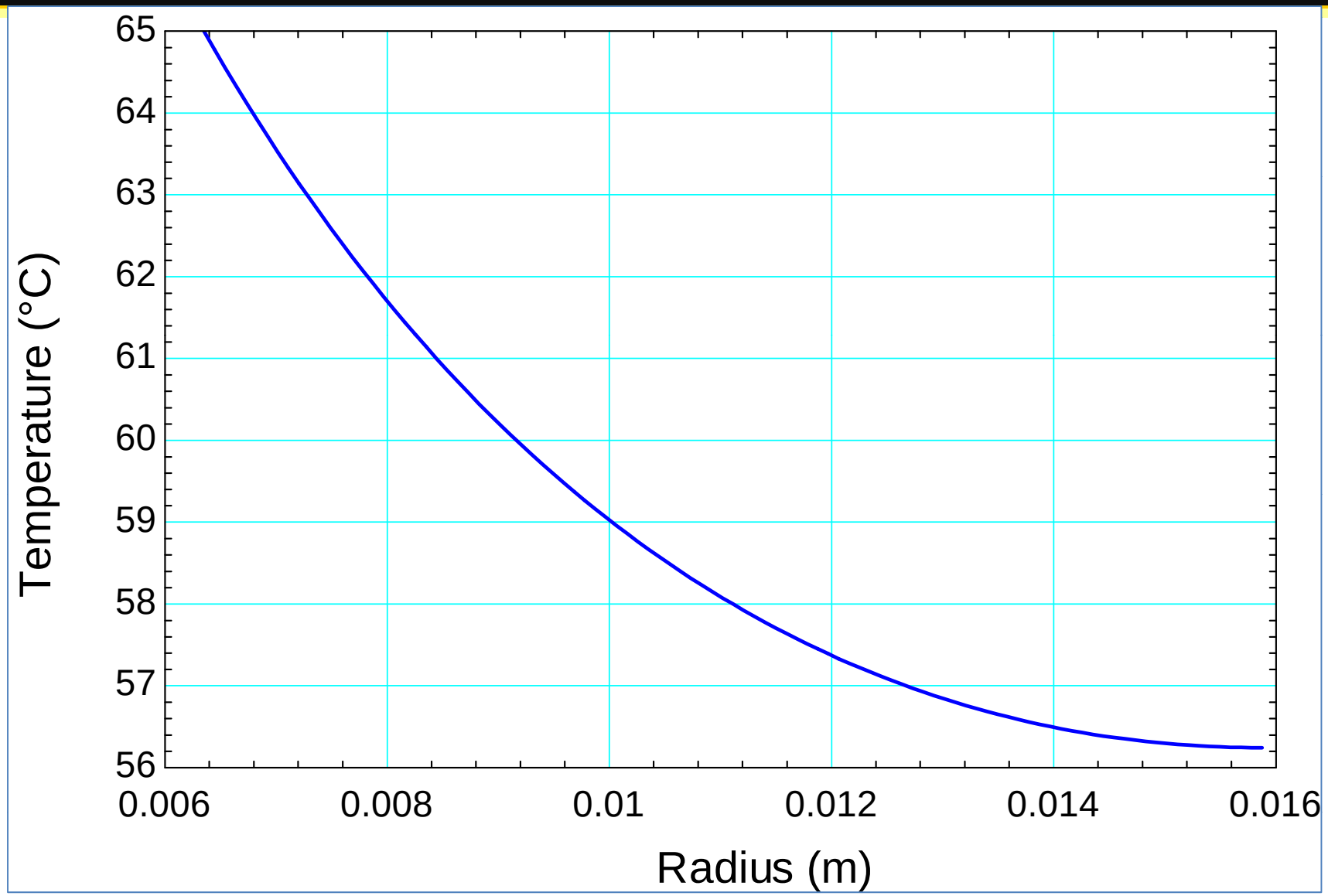
```
m=sqrt(2*h_bar/(k*th))                          "fin constant"

"Boundary conditions"
T_b=T_infinity+C_1*Bessel(0,m*r_in)+C_2*BesselK(0,m*r_in) "base temperature"
C_1*m*Bessel(1,m*r_out)-C_2*m*BesselK(1,m*r_out)=0 "adiabatic tip"
```

Boundary conditions

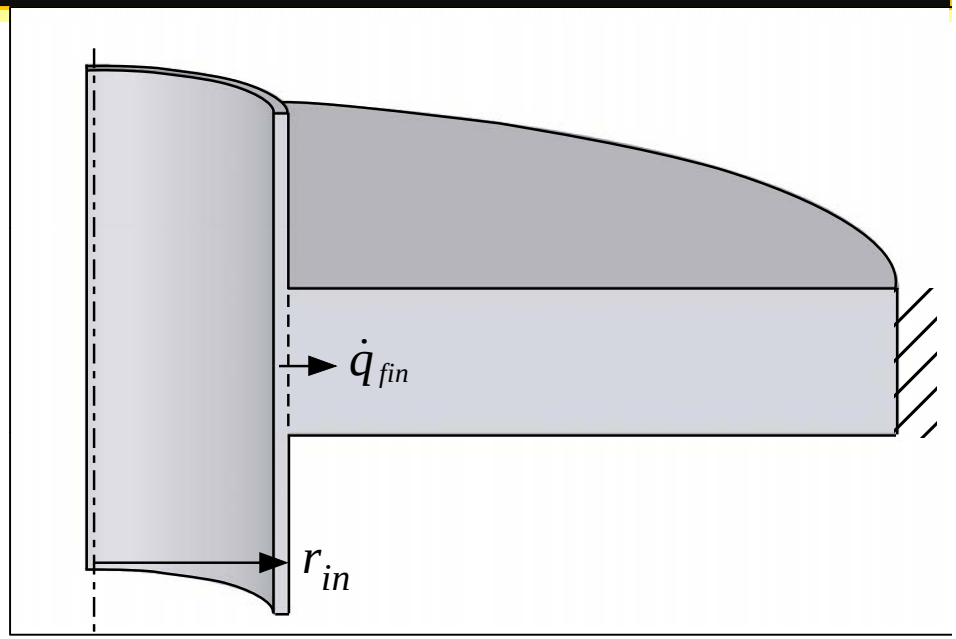
```
r_bar=0 [-]
r_bar=(r-r_in)/(r_out-r_in)                     "dimensionless radius"
T=T_infinity+C_1*Bessel(0,m*r)+C_2*BesselK(0,m*r) "temperature distribution"
T_C=converttemp(K,C,T)                          "in C"
```

General solution

EES Solution



Fin Heat Transfer



$$\dot{q}_{fin} = -2\pi r_{in}\, thk \left(\frac{dT}{dr}\right)_{r=r_{in}}$$

$$\dot{q}_{fin} = -2\pi r_{in}\, thk \frac{d}{dr}\left[C_1 \text{BesselI}(0,mr) + C_2 \text{BesselK}(0,mr) + T_\infty\right]_{r=r_{in}}$$

$$\dot{q}_{fin} = -\pi r_{in}\, thk \left\{ C_1 \underbrace{\frac{d}{dr}\left[\text{BesselI}(0,mr)\right]_{r=r_{in}}}_{m\text{BesselI}(1,mr_{in})} + C_2 \underbrace{\frac{d}{dr}\left[\text{BesselK}(0,mr)\right]_{r=r_{in}}}_{-m\text{BesselK}(1,mr_{in})} \right\}$$

$$\dot{q}_{fin} = -\pi r_{in}\, thk \left[C_1\, m\text{BesselI}(1,mr_{in}) - C_2\, m\text{BesselK}(1,mr_{in})\right]$$

Fin Efficiency

$$\eta_{fin} = \frac{\dot{q}_{fin}}{\underbrace{2\pi\left(r_{out}^2 - r_{in}^2\right)}_{A_s}\bar{h}\left(T_b - T_\infty\right)}$$

```
q_dot_fin= -2*pi*r_in*th*k*(C_1*m*Bessell(1,m*r_in)-C_2*m*BesselK(1,m*r_in))
   "heat transfer rate to fin"
eta_fin=q_dot_fin/(2*pi*(r_out^2-r_in^2)*h_bar*(T_b-T_infinity)) "fin efficiency"
eta_fin_EES=eta_fin_annular_rect(th, r_in, r_out, h_bar, k) "check EES' function"
```

we can compare our result with the internal EES function from the fin efficiency library

η_{fin} = 0.8501 [-] $\eta_{fin,EES}$ = 0.8501 [-]

Intermediate Heat Transfer
ME6300

Module 5
Separation of Variables

Lecture notes based on:

1. G. Nellis and S. A. Klein (2009), Heat Transfer, Cambridge University Press
   Lecture notes provided by Drs. Nellis and Klein

   Other textbooks such as F. P. Incropera *et al*. (2006), Fundamentals of Heat and Mass Transfer, 6th Ed, Wiley and Sons.

Requirements for Separation of Variables

- Split PDE into two ODEs that can solved using techniques to solve 1-D conduction problems
- Requirements for using Separation of Variables
  - PDE must be linear
    - cannot contain any products of temperature or its derivative

      nonlinear: $T\frac{\partial^2 T}{\partial x^2}+\frac{\partial T}{\partial x}\frac{\partial^2 T}{\partial y^2}=0$

      linear: $\frac{\partial^2 T}{\partial x^2}+\frac{\partial T}{\partial x}+\frac{\partial^2 T}{\partial x^2}+T=0$
  - PDE must be homogeneous
    - if T solution, then C T also solution (where C is a constant)

      not homogeneous: $\frac{\partial^2 T}{\partial x^2}+\frac{\partial T}{\partial x}+\frac{\partial^2 T}{\partial x^2}+T=T_\infty$

      homogeneous: $\frac{\partial^2 T}{\partial x^2}+\frac{\partial T}{\partial x}+\frac{\partial^2 T}{\partial x^2}+T=0$

Requirements for Separation of Variables

- All BCs must be linear
  - cannot contain any products of temperature or its derivative

nonlinear: $-k\left.\frac{\partial T}{\partial x}\right|_{x=W} = \sigma\,\varepsilon\,T_{x=W}^4$

linear: $-k\left.\frac{\partial T}{\partial x}\right|_{x=W} = h\left(T_{x=W} - T_\infty\right)$

- Both BCs in one direction (the ***homogeneous direction***) must be homogeneous
  - if T satisfies BCs, then C T must also satisfy BCs
  - BCs in other direction do not have to be homogeneous

not homogeneous: $-k\left.\frac{\partial T}{\partial x}\right|_{x=W} = h\left(T_{x=W} - T_\infty\right)$

homogeneous: $-k\left.\frac{\partial T}{\partial x}\right|_{x=W} = h\,T_{x=W}$

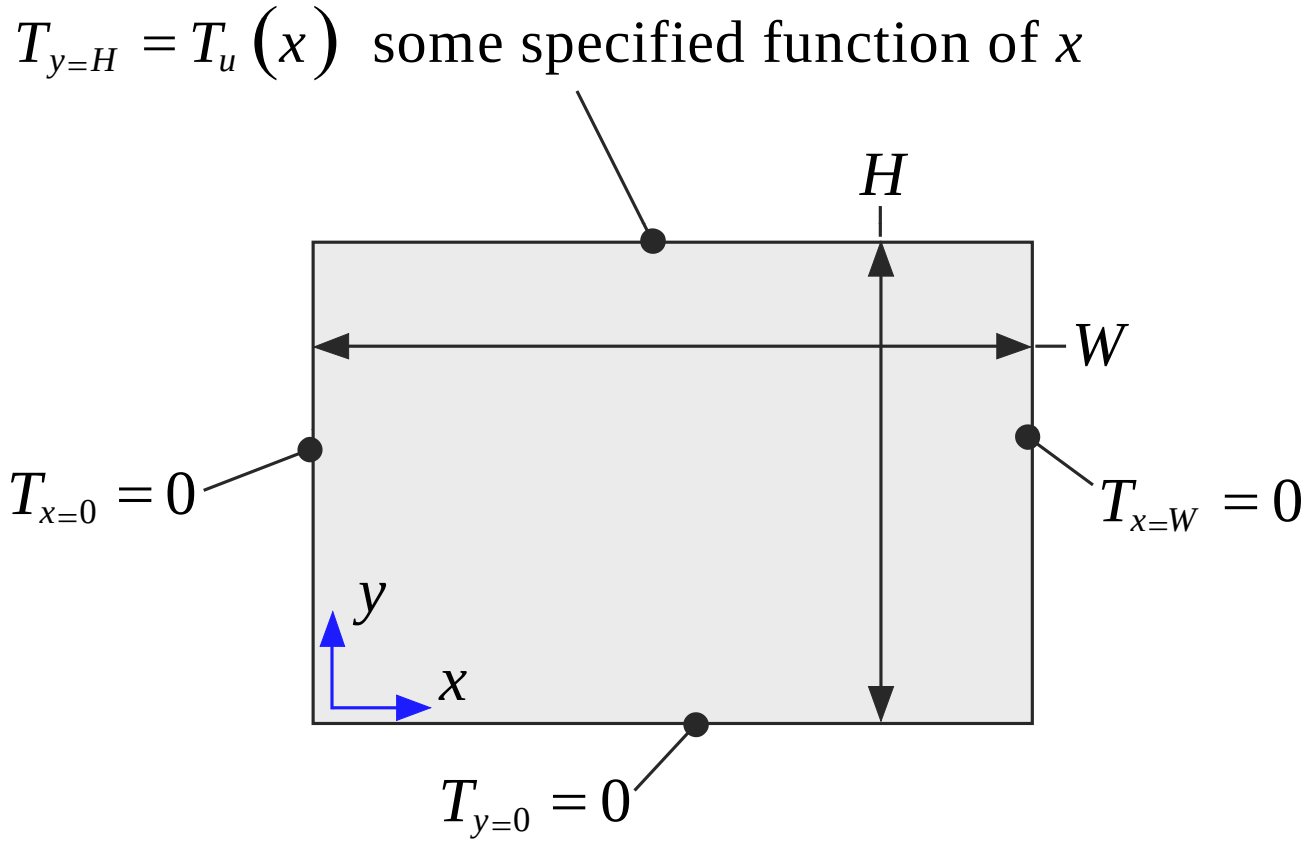
- Computational domain must be "simple"
  - boundaries must lie along lines of constant coordinates (e.g., x = W)

Separation of Variables Steps

1. Ensure that problem satisfies requirements
    1. Typically necessary to use techniques such as transformations, superposition, etc. to achieve this
    2. note which direction is "homogeneous direction"
2. Separate the variables
    1. breaks PDE into two ODEs
    2. be sure that you get the "right" ODE in the homogeneous direction
3. Solve the ***eigenproblem***
    1. Solve ODE subject to BCs in homogeneous direction
    2. provides an infinite number of ***eigenfunctions*** and ***eigenvalues***
4. Solve problem in non-homogeneous direction
5. Obtain solution for every eigenvalue
6. Assemble a series solution
    1. series solution should satisfy PDE as well as both BCs in homogeneous direction
    2. worth checking that this is true using software such as Maple
7. Enforce BCs in non-homogeneous direction
    1. Requires use of property of ***orthogonality*** of eigenfuctions at one or both BCs

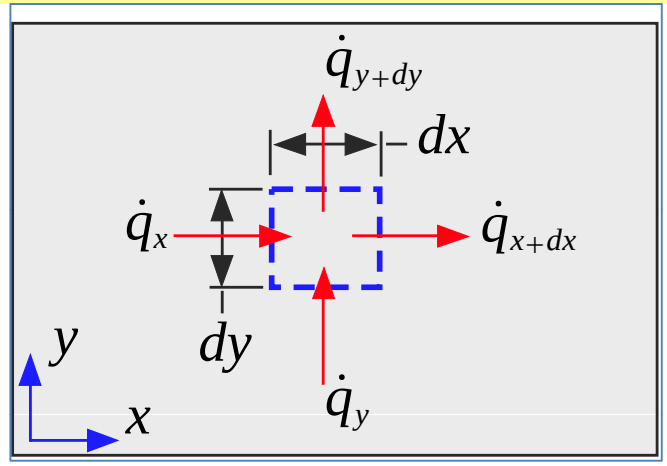
Separation of Variables Example

- Two-dimensional temperature distribution in a plate



Derive PDE

- Step 1: Differential CV
- Step 2: Energy terms
- Step 3: Energy balance



$$\dot{q}_x + \dot{q}_y = \dot{q}_{x+dx} + \dot{q}_{y+dy}$$

- Step 4: Take limit as dx and dy approach zero

$$\dot{q}_x + \dot{q}_y = \dot{q}_x + \frac{\partial \dot{q}_x}{\partial x} dx + \dot{q}_y + \frac{\partial \dot{q}_y}{\partial y} dy$$

$$0 = \frac{\partial \dot{q}_x}{\partial x} dx + \frac{\partial \dot{q}_y}{\partial y} dy$$

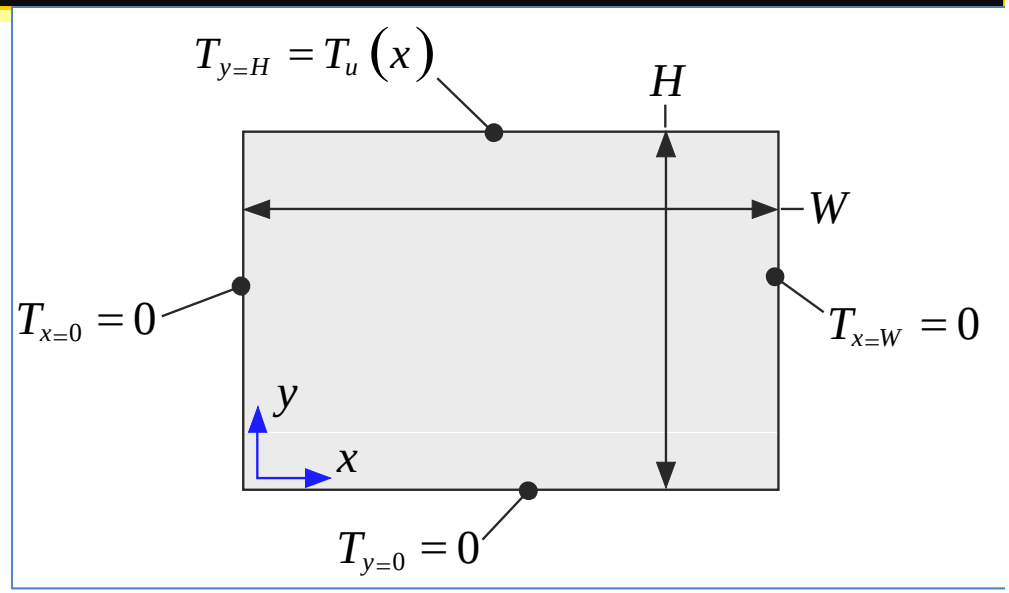
- Step 5: Rate equations

$$\dot{q}_x = -k \underbrace{L\,dy}_{\text{area for conduction}} \frac{\partial T}{\partial x} \qquad \dot{q}_y = -k\,L\,dx \frac{\partial T}{\partial y}$$

$$0 = \frac{\partial}{\partial x}\left(-k\,L\,dy \frac{\partial T}{\partial x}\right) dx + \frac{\partial}{\partial y}\left(-k\,L\,dx \frac{\partial T}{\partial y}\right) dy \qquad \boxed{0 = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}}$$

Mathematical Specification of Problem

- Requirements for SOV:
  - PDE is linear
  - PDE is HG
  - BCs are linear
  - BCs in one direction are both HG
  - Computational domain is simple



$$0 = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

$$T_{x=0} = 0$$

$$T_{x=W} = 0$$

$$T_{y=0} = 0$$

$$T_{y=H} = T_u(x)$$

x is the homogeneous direction for this problem

Separate the Variables

- Assume that the solution, $T(x,y)$, can be expressed as the product of two functions: $Tx(x)$ and $Ty(y)$:

$$T(x,y) = TX(x)TY(y)$$

- Substitute into PDE:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{\partial^2}{\partial x^2}[TX\,TY] + \frac{\partial^2}{\partial y^2}[TX\,TY] = 0$$

$$TY\frac{d^2TX}{dx^2} + TX\frac{d^2TY}{dy^2} = 0$$

- Divide through by $TY\ TX$:

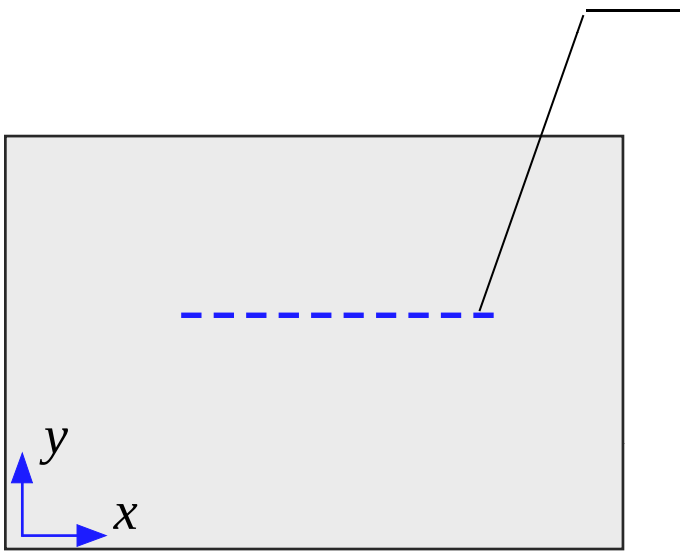
$$\frac{TY\dfrac{d^2TX}{dx^2}}{TX\,TY} + \frac{TX\dfrac{d^2TY}{dy^2}}{TX\,TY} = 0$$

$$\boxed{\frac{\dfrac{d^2TX}{dx^2}}{TX} + \frac{\dfrac{d^2TY}{dy^2}}{TY} = 0}$$

Separate the Variables

- Equation can only be satisfied if both terms:
  1. have the same magnitude
  2. have opposite sign
  3. are constant

$$\underbrace{\frac{\frac{d^2TX}{dx^2}}{TX}}_{\text{function of } x} + \underbrace{\frac{\frac{d^2TY}{dy^2}}{TY}}_{\text{function of } y} = 0$$



If these terms were not constant, then $\frac{\frac{d^2TX}{dx^2}}{TX}$ would change as I move along a line of constant x

but $\frac{\frac{d^2TY}{dy^2}}{TY}$ could not change in response...

Split into ODEs

- The sign of the constant that you choose matters!

$$\frac{\frac{d^2TX}{dx^2}}{TX} = \pm\lambda^2 \qquad \frac{\frac{d^2TY}{dy^2}}{TY} = \mp\lambda^2$$

positive or negative?

- The choice of sign leads to the form of the ODE:

$$\frac{d^2TX}{dx^2} - \lambda^2\, TX = 0 \;\text{ and }\; \frac{d^2TY}{dy^2} + \lambda^2\, TY = 0 \longrightarrow$$

TX is solved by sinh and cosh
TY is solved by sin and cos

$$\frac{d^2TX}{dx^2} + \lambda^2\, TX = 0 \;\text{ and }\; \frac{d^2TY}{dy^2} - \lambda^2\, TY = 0 \longrightarrow$$

TX is solved by sin and cos
TY is solved by sinh and cosh

- The homogeneous direction (x, for this problem) leads to our eigenproblem - the eigenfunctions must be sin and cos (not sinh and cosh)...

Split into ODEs

- We have successfully split our PDE into two ODES:

$$\frac{d^2TX}{dx^2} + \lambda^2 TX = 0 \text{ and } \frac{d^2TY}{dy^2} - \lambda^2 TY = 0$$

homogeneous direction - this is our <u>*eigenproblem*</u>

- Solve the ***eigenproblem:***
  - recognize that the solution to this particular HG ODE is sine and cosine...

$$TX = C_1 \sin(\lambda x) + C_2 \cos(\lambda x)$$

Eigen Problem

- Boundary conditions:
  - at x = 0: $T_{x=0} = 0 \rightarrow TX_{x=0}\, TY = 0$
    - is satisfied if TY = 0 (not a useful solution) or if: $TX_{x=0} = 0$
    - substitute general solution:

$$TX_{x=0} = C_1 \underbrace{\sin(\lambda 0)}_{=0} + C_2 \underbrace{\cos(\lambda 0)}_{=1} = 0$$

$$C_2 = 0$$

$$TX = C_1 \sin(\lambda x)$$

Eigenfunctions and Eigenvalues

- Boundary conditions:
  - at x = W: $T_{x=W} = 0 \rightarrow TX_{x=W}\, TY = 0$
    - again, this is satisfied if TY = 0 (not a useful solution) or if: $TX_{x=W} = 0$
    - substitute general solution: $TX_{x=W} = C_1 \sin(\lambda W) = 0$
    - this is satisfied if C_1 = 0 (not a useful solution) or if: $\sin(\lambda W) = 0$
    - this occurs if: $\lambda_i = \frac{i\pi}{W}$ for $i = 1 \ldots \infty$
- Infinite number of solutions to the eigenproblem:

$$\boxed{TX_i = C_{1,i} \underbrace{\sin(\lambda_i x)}_{\text{eigenfunction}} \text{ where } \underbrace{\lambda_i = \frac{i\pi}{W}}_{\text{eigenvalue}} \text{ for } i = 1 \ldots \infty}$$

Solve the Nonhomogeneous Problem

- There is an ODE for TY (i.e., the nonhomogeneous direction) associated with each eigenvalue:

$$\frac{d^2 TY_i}{dy^2} - \lambda_i^2 \, TY_i = 0$$

  – recognize that this HG ODE is solved by sinh and cosh

$$TY_i = C_{3,i} \sinh\left(\lambda_i \, y\right) + C_{4,i} \cosh\left(\lambda_i \, y\right)$$

Obtain a Solution for Each Eigenvalue

- Infinite solutions for TX (TX_i for i = 1…∞).
- For each solution Tx_i, an associated solution Ty_i
- Solution for T associated with each eigenvalue is:

$$T_i = TX_i\, TY_i$$

$$T_i = C_{1,i} \sin(\lambda_i x)\left[C_{3,i} \sinh(\lambda_i y) + C_{4,i} \cosh(\lambda_i y)\right]$$

- Product of two undetermined constants is an undetermined constant (i.e., it makes no sense to keep $C_{1,i}$, $C_{3,i}$, and $C_{4,i}$):

$$T_i = \sin(\lambda_i x)\left[\underbrace{C_{1,i}\, C_{3,i}}_{C_{3,i}} \sinh(\lambda_i y) + \underbrace{C_{1,i}\, C_{4,i}}_{C_{4,i}} \cosh(\lambda_i y)\right]$$

$$\boxed{T_i = \sin(\lambda_i x)\left[C_{3,i} \sinh(\lambda_i y) + C_{4,i} \cosh(\lambda_i y)\right]}$$

Assemble the Series Solution

- Each solution T_i satisfies PDE and HG-direction BCs
- Because PDE and BCs are linear and HG, sum of all solutions also satisfies PDE and BCs:

$$T = \sum_{i=1}^{\infty} T_i$$

$$T = \sum_{i=1}^{\infty} \sin(\lambda_i x)\left[C_{3,i} \sinh(\lambda_i y) + C_{4,i} \cosh(\lambda_i y)\right]$$

Enforce the Nonhomogeneous BCs

- At $y = 0$

$$T_{y=0} = 0$$

$$T_{y=0} = \sum_{i=1}^{\infty} \sin(\lambda_i x)\left[C_{3,i}\underbrace{\sinh(\lambda_i 0)}_{0} + C_{4,i}\underbrace{\cosh(\lambda_i 0)}_{1}\right] = 0$$

$$\sum_{i=1}^{\infty} C_{4,i} \sin(\lambda_i x) = 0$$

- This only works if $C_{4,i} = 0$ for all i:

$$T = \sum_{i=1}^{\infty} C_{3,i} \sin(\lambda_i x)\sinh(\lambda_i y)$$

- Since there is only one constant left, it is not necessary to label it C_3

$$T = \sum_{i=1}^{\infty} C_i \sin(\lambda_i x)\sinh(\lambda_i y)$$

Enforce the Nonhomogeneous BCs

- At $y = H$ $$T_{y=H} = T_u(x)$$

$$\sum_{i=1}^{\infty} C_i \sin(\lambda_i x)\sinh(\lambda_i H) = T_u(x)$$

- This equation defines all constants in solution
- They are related to Fourier coefficients of non-homogeneous BC
- At first glance, it's not clear how we can actually use this equation (which implicitly relates all of the constants) to explicitly determine each constant
- We have to use a special property of the eigenfunctions - they are <u>*orthogonal*</u>

Orthogonality of Eigenfunctions

- What does it mean for a set of functions, $F_1(x)$ to $F_N(x)$, to be orthogonal over a certain range ($x_{start} < x < x_{end}$)?
  - If two different functions in the set are multiplied and integrated over the range the result will be zero

$$\int_{x_{start}}^{x_{end}} F_i(x) F_j(x) dx = \begin{cases} 0 \text{ if } i \neq j \\ \neq 0 \text{ if } i = j \end{cases}$$

- Eigenfunctions of our SOV solution are guaranteed to be orthogonal when integrated from one boundary to the other in the homogeneous direction
- For this problem: $\int_0^w \sin(\lambda_i x) \sin(\lambda_j x) dx = \begin{cases} 0 \text{ if } i \neq j \\ \neq 0 \text{ if } i = j \end{cases}$

Using the Orthogonality of Eigenfunctions

$$\sum_{i=1}^{\infty} C_i \sin(\lambda_i x)\sinh(\lambda_i H) = T_u(x)$$

- Multiply both sides of equation by an eigenfunction

$$\sum_{i=1}^{\infty} C_i \sin(\lambda_i x)\sinh(\lambda_i H)\sin(\lambda_i x) = T_u(x)\sin(\lambda_i x)$$

- and integrate from $x = 0$ to $x = W$

$$\int_0^W \sum_{i=1}^{\infty} C_i \sin(\lambda_i x)\sinh(\lambda_i H)\sin(\lambda_j x)dx = \int_0^W T_u(x)\sin(\lambda_j x)dx$$

$$\sum_{i=1}^{\infty} C_i \sinh(\lambda_i H)\boxed{\int_0^W \sin(\lambda_i x)\sin(\lambda_j x)dx} = \int_0^W T_u(x)\sin(\lambda_j x)dx$$

due to the orthogonality of eigenfunctions, every one of these terms must be zero except the one term where $i = j$

Determine the Constants

$$C_j \sinh\left(\lambda_j H\right)\int_0^W \sin^2\left(\lambda_j x\right)dx = \int_0^W T_u\left(x\right)\sin\left(\lambda_j x\right)dx$$

- We have turned a series equation into an explicit equation for each of the constants:

$$C_j = \frac{\boxed{\int_0^W T_u\left(x\right)\sin\left(\lambda_j x\right)dx}}{\sinh\left(\lambda_j H\right)\boxed{\int_0^W \sin^2\left(\lambda_j x\right)dx}}$$

these integrals can be evaluated conveniently using Maple

```
> int((sin(lambda_j*x))^2,x=0..W);
```

$$\frac{W}{2}$$

$$\longrightarrow \quad C_j = \frac{2\int_0^W T_u\left(x\right)\sin\left(\lambda_j x\right)dx}{W\sinh\left(\lambda_j H\right)}$$

Determine the Constants

- Solution for constant T on upper surface:

$$T_u(X) = T_u$$

$$C_j = \frac{2\boxed{\int_0^W T_u \sin(\lambda_j x)\,dx}}{W\sinh(\lambda_j H)}$$

```
> int(T_u*sin(lambda_j*x),x=0..W);
```

$$-\frac{T\_u\, W\left(-1+(-1)^{j\sim}\right)}{j\sim\,\pi}$$

$$C_j = -\frac{2T_u\left[-1+(-1)^j\right]}{j\pi\sinh(\lambda_j H)}$$

this equation provides the jth constant

- So, the solution is:

$$\boxed{T = \sum_{i=1}^{\infty} C_i \sin(\lambda_i x)\sinh(\lambda_i y) \text{ where } C_i = -\frac{2T_u\left[-1+(-1)^i\right]}{i\pi\sinh(\lambda_i H)}}$$

Implement Solution in EES

```
$UnitSystem SI, Mass, Radian, J, K, Pa
$TabStops 0.25, 3 in
```

note that this is important for these problems

```
W=1 [m]                                   "width of plate"
H=1 [m]                                   "height of plate"
T_u=1 [K]                                 "temperature of upper surface"
```

inputs

```
N=10 [-]                                  "number of terms"
```

how many terms are required?

```
duplicate i=1,N
    lambda[i]=i*pi/W                      "i'th eigenvalue"
    C[i]=-2*T_u*(-1+(-1)^i)/(i*pi*sinh(lambda[i]*H)) "i'th constant"
end
```

evaluate eigenvalues and constants for each of the *N* terms

```
x=0.5 [m]                                 "x-position to obtain solution"
y=0.5 [m]                                 "y-position to obtain solution"
```

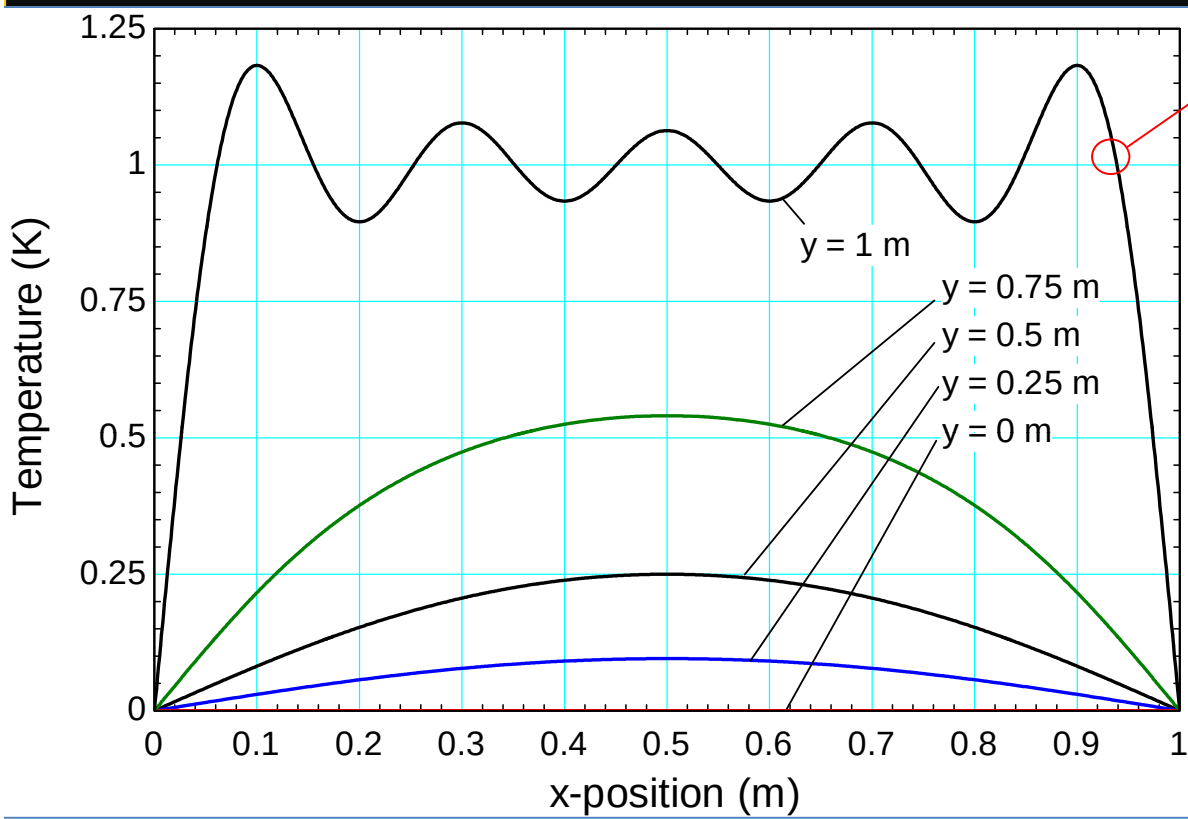
```
duplicate i=1,N
    T[i]=C[i]*sin(lambda[i]*x)*sinh(lambda[i]*y) "solution for i'th term"
end
```

evaluate each term of the series

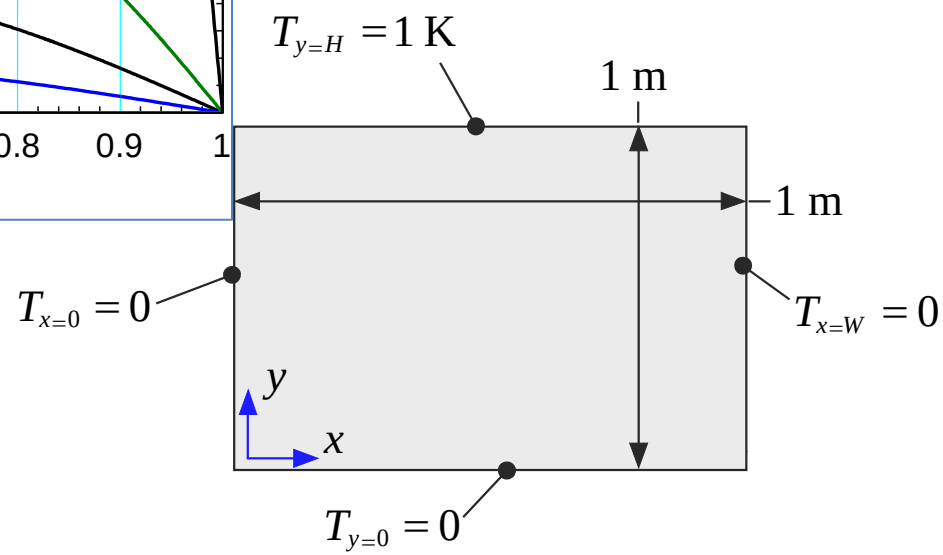
```
T=sum(T[i],i=1,N)                         "solution"
```

sum all of the terms

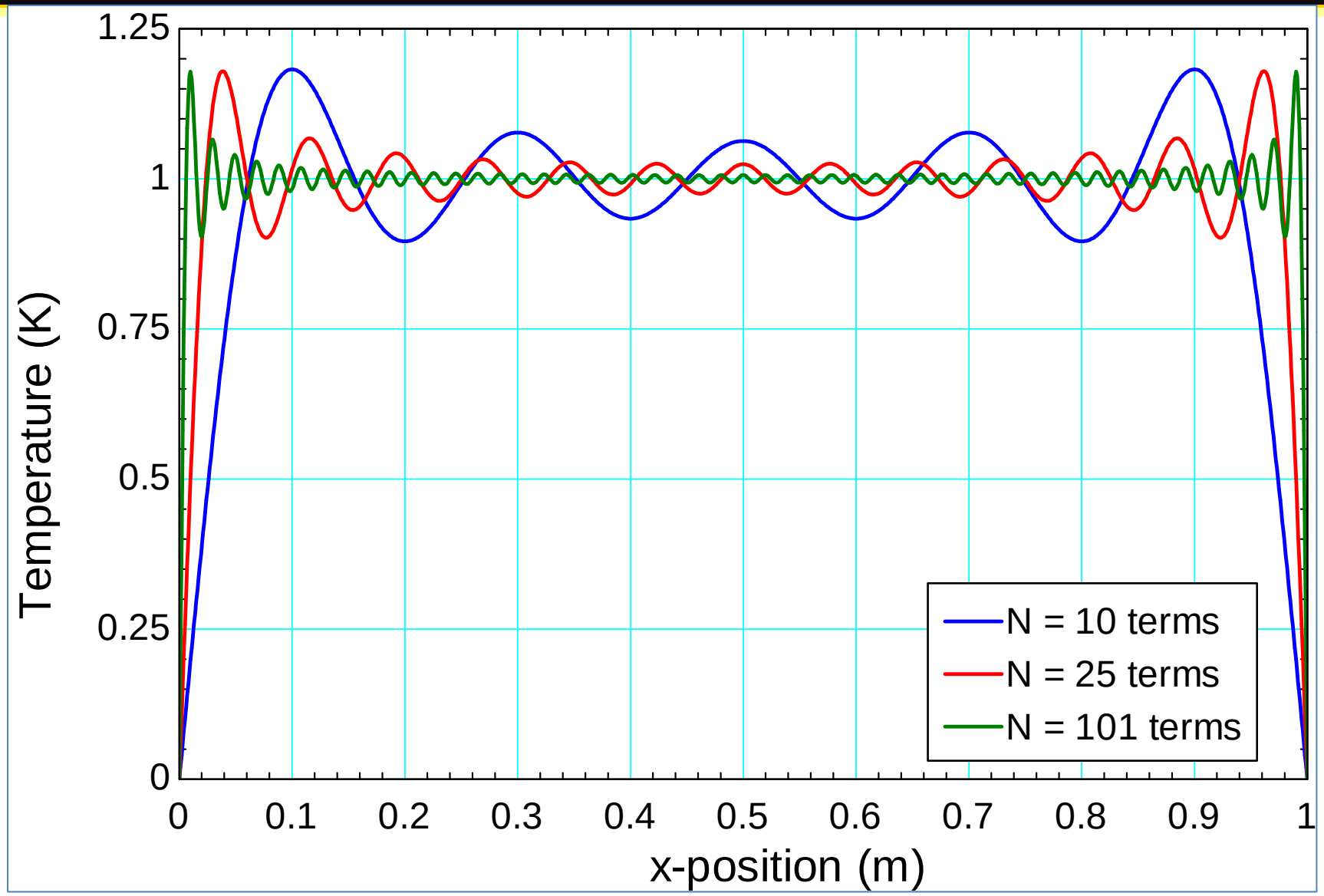
Solution in EES



notice that 10 terms are not sufficient to match the imposed boundary condition at $y = H$



Number of Terms



How Many Terms are Enough?

Error in terminating the series is bounded by magnitude of last nonzero term

N = 11, x = 0.5 m, y = 0.5 m
(center of plate)

Arrays Table

Sort	C_i [K]	λ_i [1/m]	T_i [K]
[1]	0.1102	3.142	0.2537
[2]	0	6.283	0
[3]	0.0000685	9.425	-0.003812
[4]	0	12.57	0
[5]	7.675E-08	15.71	0.0000988
[6]	0	18.85	0
[7]	1.023E-10	21.99	-0.000003
[8]	0	25.13	0
[9]	1.487E-13	28.27	1.025E-07
[10]	0	31.42	0
[11]	2.272E-16	34.56	-3.626E-09

error is small, < 4×10^{-9} K

N = 11, x = 0.5 m, y = 1 m
(center of upper edge)

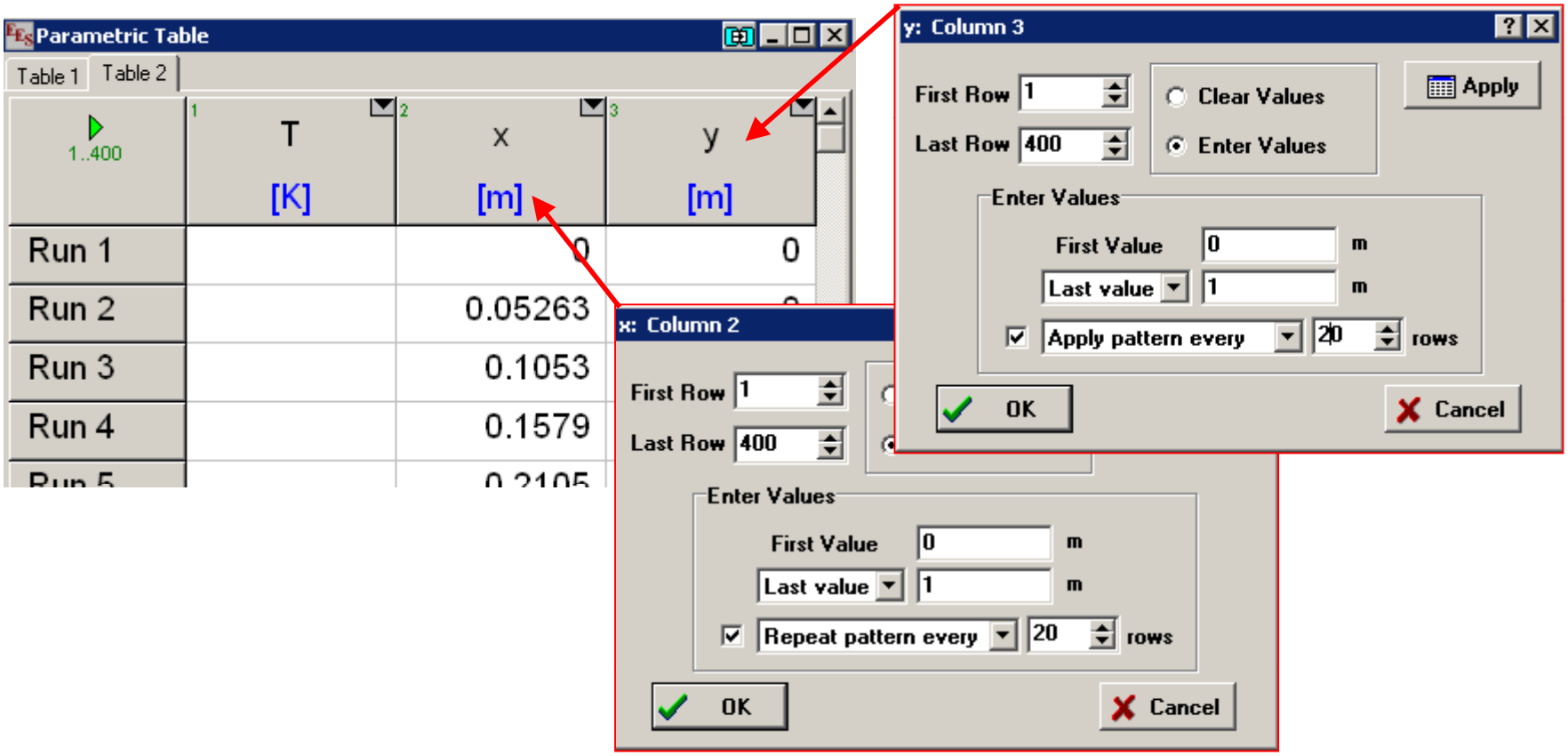
Arrays Table

Sort	C_i [K]	λ_i [1/m]	T_i [K]
[1]	0.1102	3.142	1.273
[2]	0	6.283	0
[3]	0.0000685	9.425	-0.4244
[4]	0	12.57	0
[5]	7.675E-08	15.71	0.2546
[6]	0	18.85	0
[7]	1.023E-10	21.99	-0.1819
[8]	0	25.13	0
[9]	1.487E-13	28.27	0.1415
[10]	0	31.42	0
[11]	2.272E-16	34.56	-0.1157

error is larger, < 0.12 K

Contour Plots

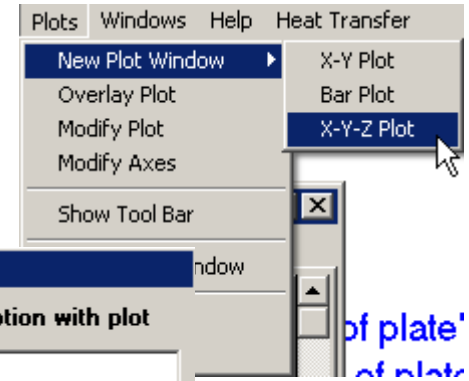
- Parametric table filled with 2-D grid of x and y
  - 400 runs for a 20x20 grid
  - repeat pattern every 20 rows in x
  - apply pattern every 20 rows in y

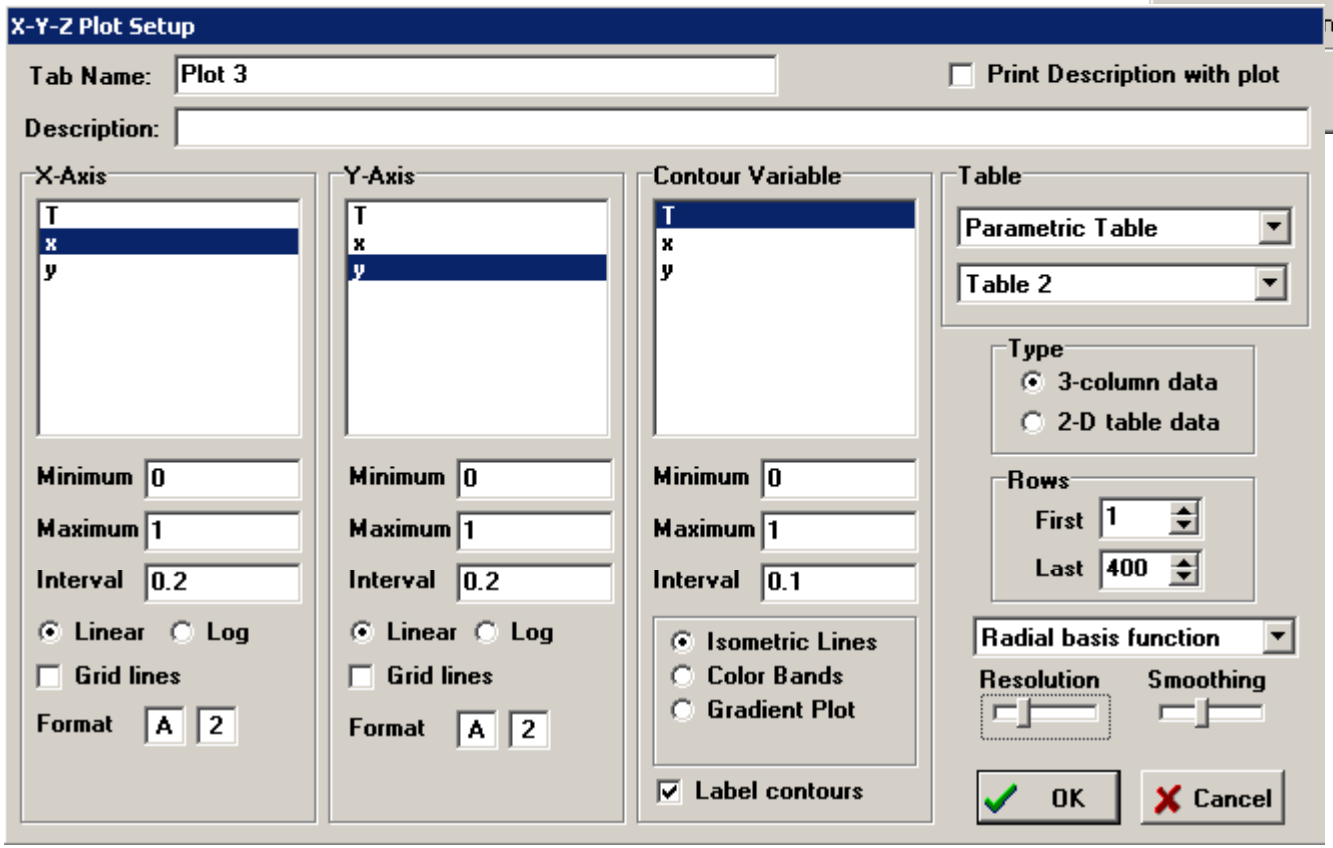


Contour Plots

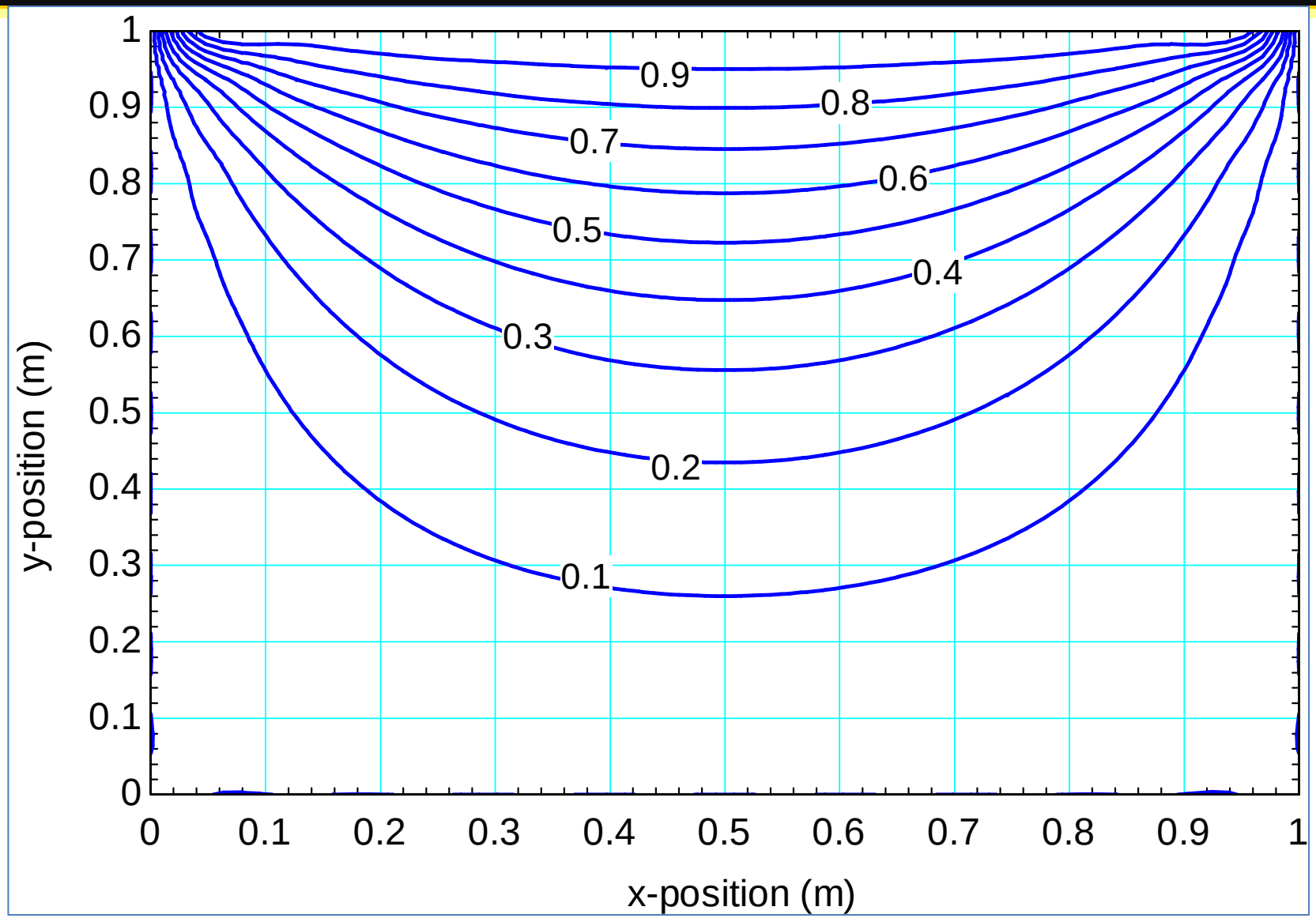
Select X-Y-Z Plot from the Plots menu

Select correct table and setup figure

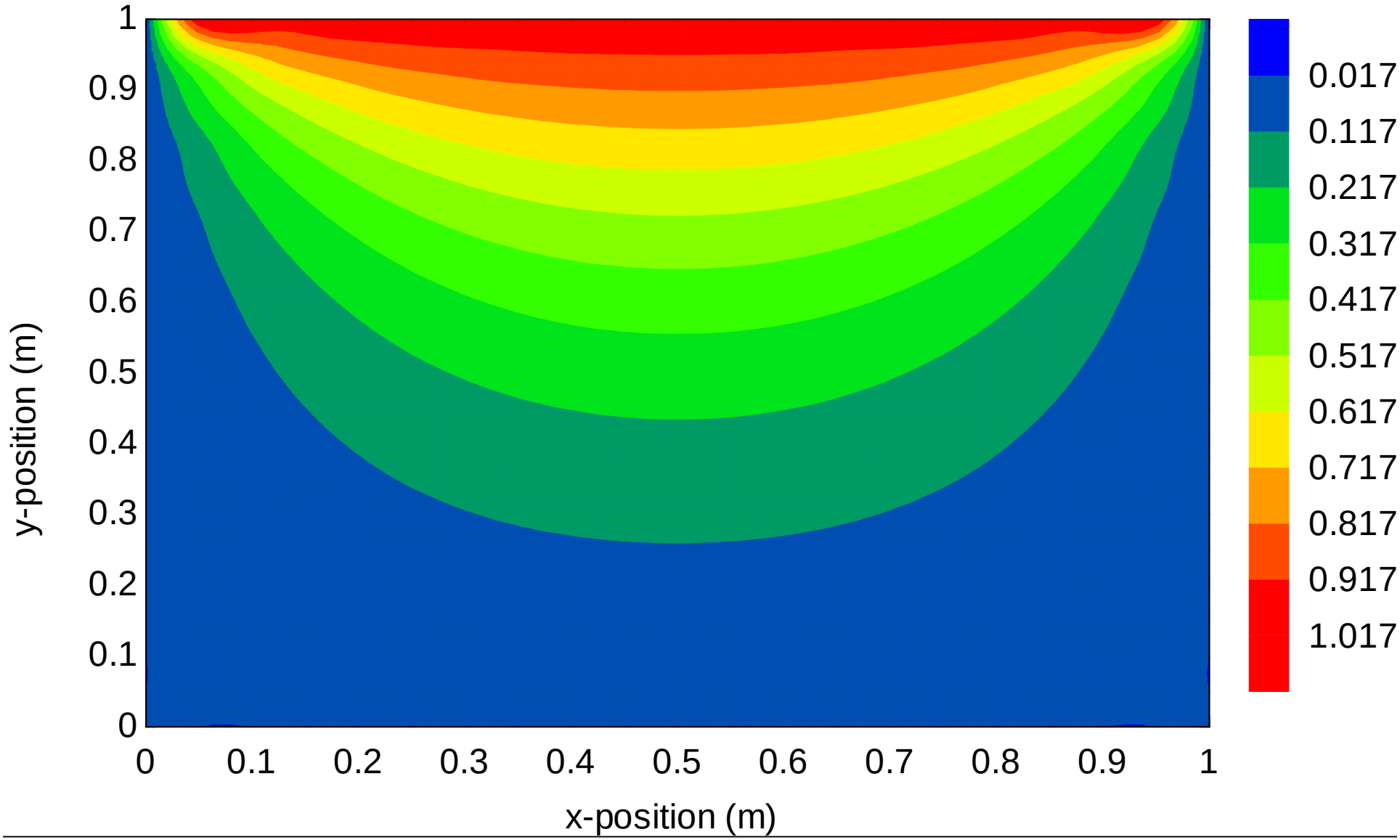




Contour Plots



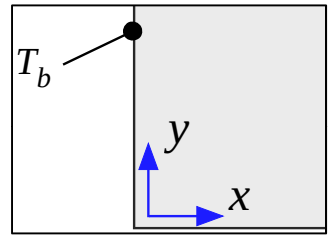
Contour Plots



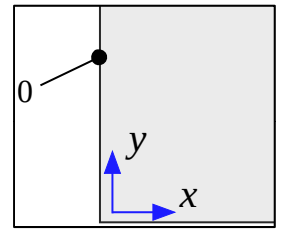
Homogeneous BCs

Three types of linear BCs - each has homogeneous equivalent:

- Specified temperature:

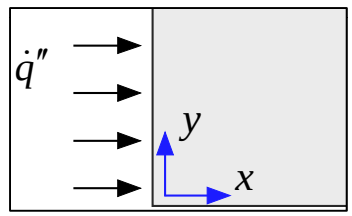


$T_{x=0} = T_b$

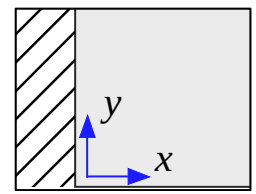


$T_{x=0} = 0$ (homogeneous)

- Specified heat flux:

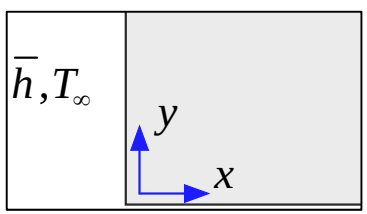


$$-k\left(\frac{\partial T}{\partial x}\right)_{x=0} = \dot{q}''$$

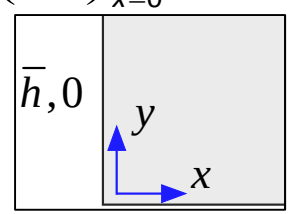


$$-k\left(\frac{\partial T}{\partial x}\right)_{x=0} = 0 \text{ (homogeneous)}$$

- Convection:



$$-k\left(\frac{\partial T}{\partial x}\right)_{x=0} = \bar{h}\left(T_\infty - T_{x=0}\right)$$



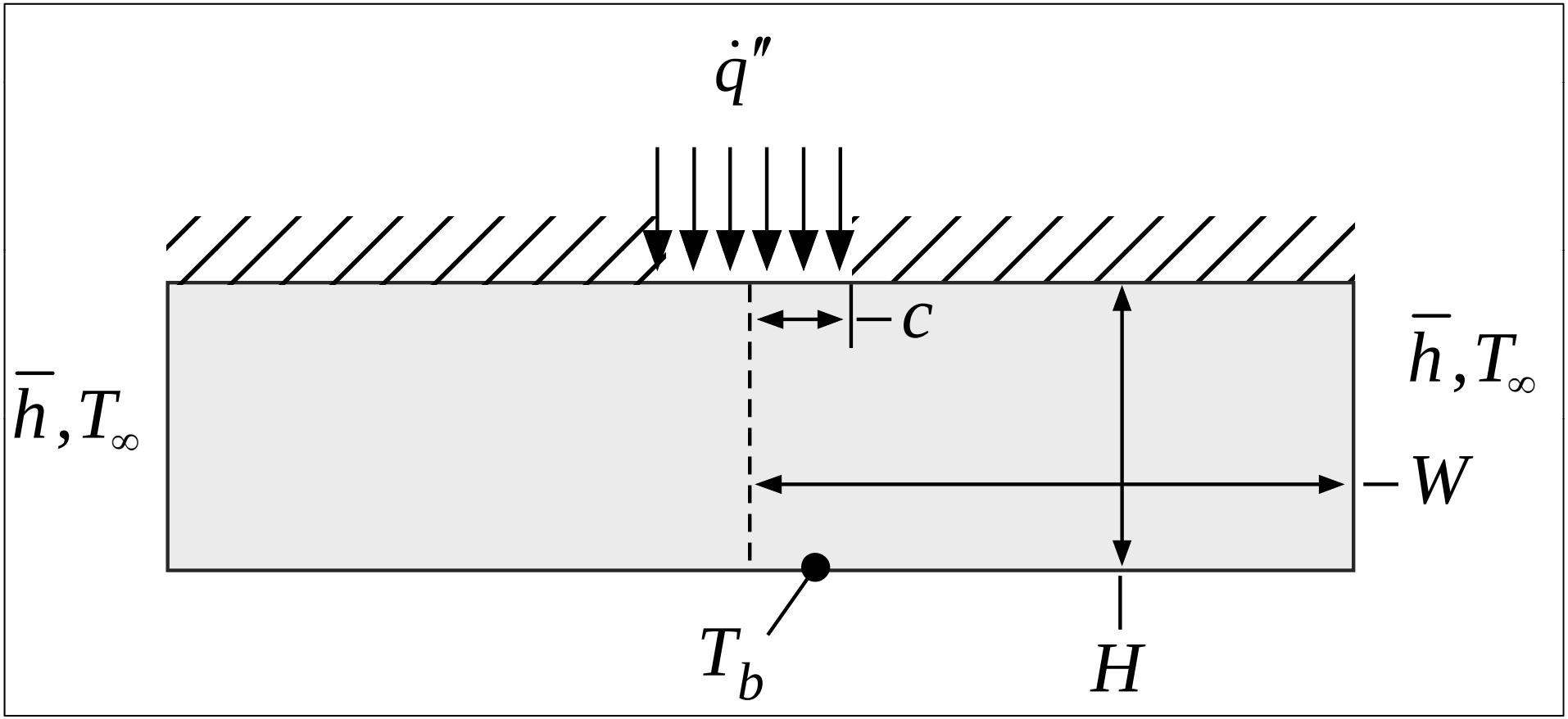
$$k\left(\frac{\partial T}{\partial x}\right)_{x=0} = \bar{h}\,T_{x=0} \text{ (homogeneous)}$$

Homogeneous BCs

- Very few problems naturally have two homogeneous boundary conditions in one direction
- In many cases, use superposition or more advanced technique (see Section 2.3) to address this
- In some cases, very simple transformation can be used to recast the problem to provide one additional homogeneous BC
  - transform problem in terms of θ, where θ is ΔT relative to one of the boundary or fluid temperatures

Example

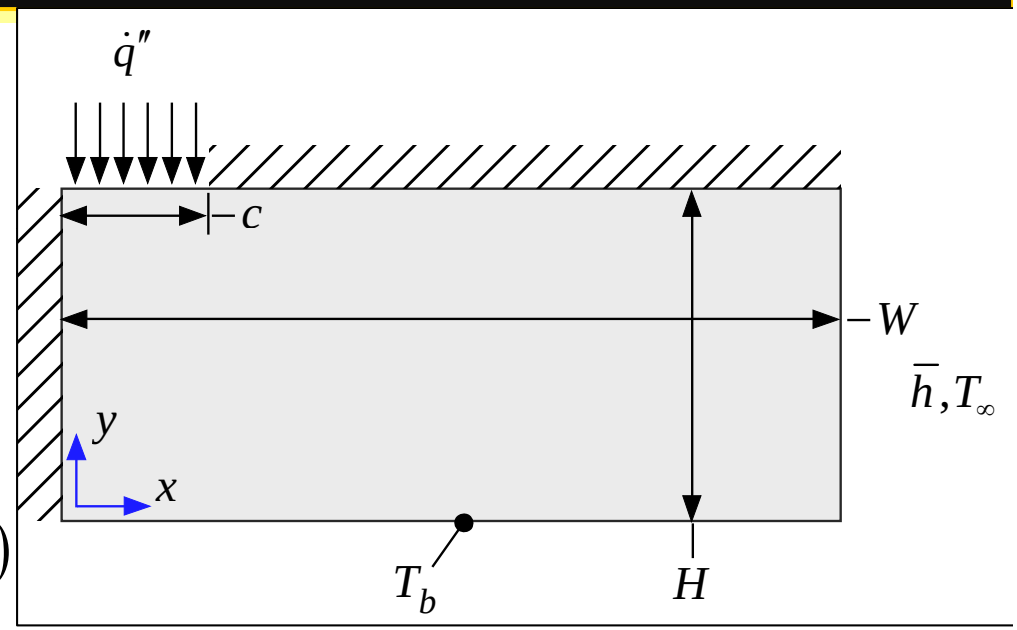
- Machining process applies a heat flux at the top center of a plate



- Develop a half-symmetry model of the process

Half-Symmetry Model

- Mathematical specification:
- PDE: $$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$
- BC:



$$\left(\frac{\partial T}{\partial x}\right)_{x=0}=0 \qquad -k\left(\frac{\partial T}{\partial x}\right)_{x=W}=\bar{h}\left(T_{x=W}-T_\infty\right)$$

$$T_{y=0}=T_b \qquad k\left(\frac{\partial T}{\partial y}\right)_{y=H}=\begin{cases}\dot{q}'' & \text{for } 0<x<c\\ 0 & \text{for } c<x<W\end{cases}$$

- Notice that neither direction has two homogeneous BCs

Transformation

- *x*-direction BC at *x* = *W* can be made homogeneous by transforming problem according to:

$$\theta = T - T_\infty$$

<u>*Transformed problem specification*</u>

- PDE: $\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} = 0$

- BC: $\left(\dfrac{\partial\theta}{\partial x}\right)_{x=0} = 0 \quad -k\left(\dfrac{\partial\theta}{\partial x}\right)_{x=W} = \bar{h}\,\theta_{x=W}$

$$\theta_{y=0} = T_b - T_\infty \quad k\left(\frac{\partial\theta}{\partial y}\right)_{y=H} = \begin{cases} \dot{q}'' & \text{for } 0 < x < c \\ 0 & \text{for } c < x < W \end{cases}$$

- Transformed problem has two homogeneous BC in x-direction

Separate the Variables

- Assume that: $\theta(x,y) = \theta X(x)\theta Y(y)$
- Substitute into PDE:

$$\theta Y \frac{d^2\theta X}{dx^2} + \theta X \frac{d^2\theta Y}{dy^2} = 0$$

$$\boxed{\frac{\frac{d^2\theta X}{dx^2}}{\theta X}} + \boxed{\frac{\frac{d^2\theta Y}{dy^2}}{\theta Y}} = 0$$

need eigenfunctions in the *x*-direction... this group must be equal to $-\lambda^2$

therefore, this group must be equal to $+\lambda^2$

- The two ODEs:

$$\frac{d^2\theta X}{dx^2} + \lambda^2\theta X = 0 \qquad \frac{d^2\theta Y}{dy^2} - \lambda^2\theta Y = 0$$

Solve the Eigenproblem

- ODE in the homogeneous direction: $\frac{d^2\theta X}{dx^2}+\lambda^2\theta X=0$
- General solution: $\theta X = C_1\cos(\lambda x)+C_2\sin(\lambda x)$
- Boundary condition at x = 0:

$$\left(\frac{\partial\theta}{\partial x}\right)_{x=0}=0 \longrightarrow \theta Y\left(\frac{d\theta X}{dx}\right)_{x=0}=0 \longrightarrow \left(\frac{d\theta X}{dx}\right)_{x=0}=0$$

$$C_1\lambda\underbrace{\sin(\lambda 0)}_{0}+C_2\lambda\underbrace{\cos(\lambda 0)}_{1}=0$$

$$C_2=0$$

$$\theta X = C_1\cos(\lambda x)$$

Eigencondition

- Boundary condition at x = W: $$-k\left(\frac{d\theta}{dx}\right)_{x=W} = \bar{h}\,\theta_{x=W}$$

$$-k\,\theta Y\left(\frac{d\theta X}{dx}\right)_{x=W} = \bar{h}\,\theta Y\,\theta X_{x=W} \longrightarrow -k\left(\frac{d\theta X}{dx}\right)_{x=W} = \bar{h}\,\theta X_{x=W}$$

$$k C_1 \lambda \sin(\lambda W) = \bar{h}\, C_1 \cos(\lambda W)$$

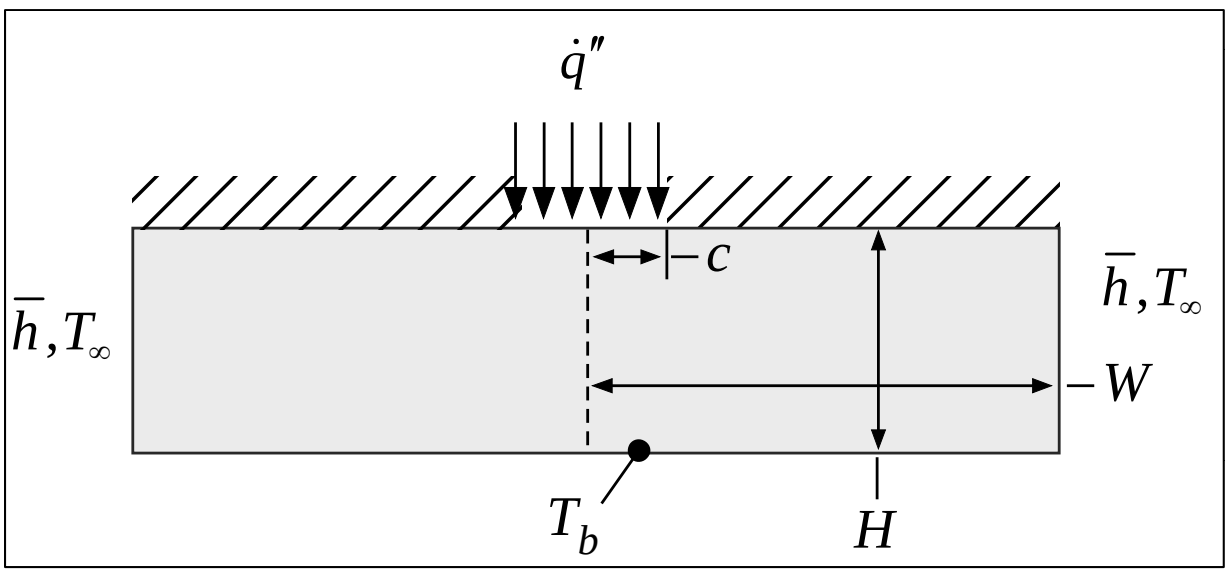
- The eigencondition for this problem: $$\boxed{\tan(\lambda W) = \frac{\bar{h}}{k\lambda}}$$

- This eigencondition:
  - defines an infinite number of eigenvalues, but
  - does not explicitly provide each eigenvalue

Implementing Solution in EES

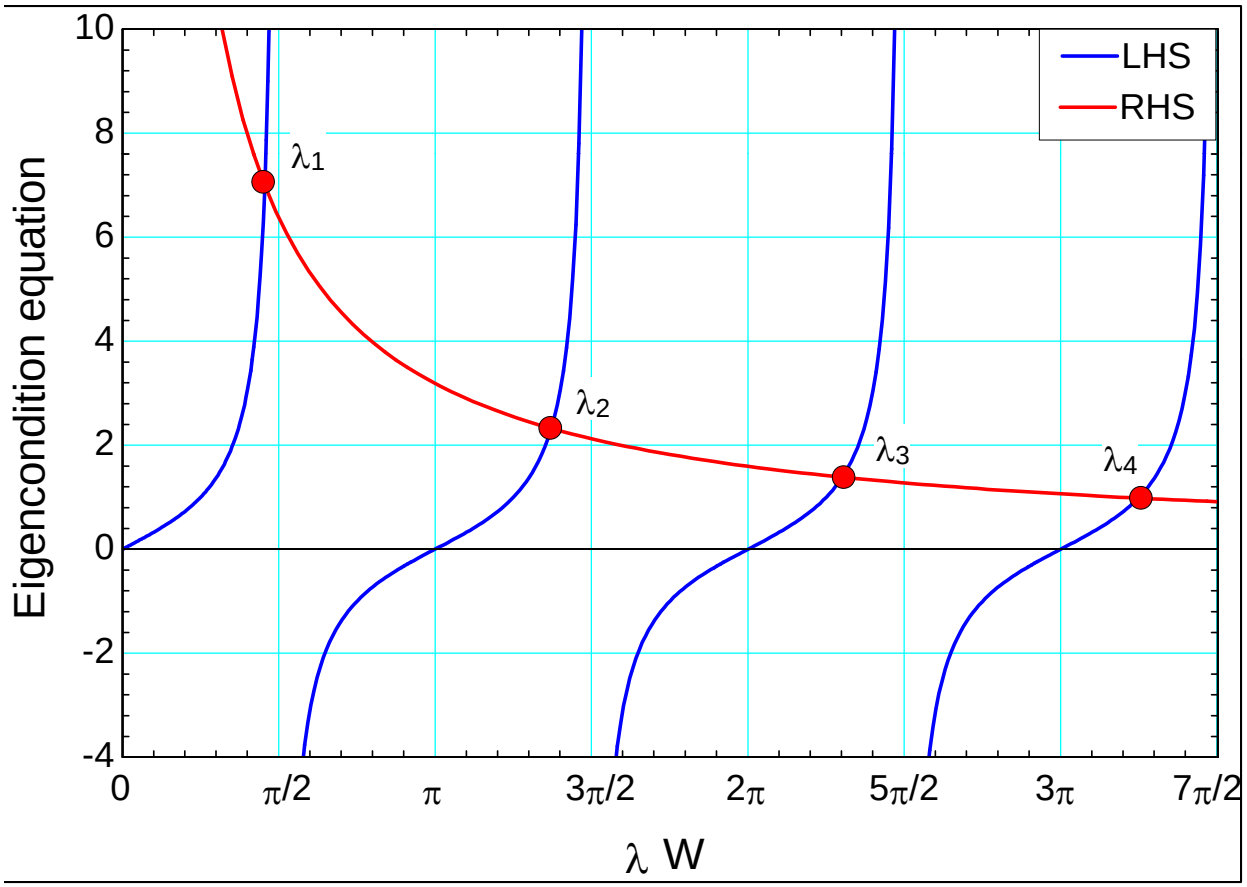
- Inputs:

```
$UnitSystem SI, Mass, Radian, J, K, Pa
$TabStops 0.25, 3 in

W=0.1 [m]                                  "half-width of plate"
H=0.05 [m]                                 "height of plate"
qf=5.4e4 [W/m^2]                           "heat flux"
c=0.015 [m]                                "half-width of heat flux region"
k=2.5 [W/m-K]                              "conductivity"
h_bar=250 [W/m^2-K]                        "average heat transfer coefficient"
T_infinity=converttemp(C,K,20 [C])         "ambient temperature"
T_b=converttemp(C,K,200 [C])               "base temperature"
```



Eigenvalue Ranges

- Eigencondition: $\underbrace{\tan(\lambda W)}_{RHS} = \underbrace{\frac{\bar{h} W}{k(\lambda W)}}_{LHS}$



Note that successive eigenvalues fall in ranges defined by argument of the trigonometric function, λW
For this problem, the ranges are:

$$0 < \lambda_1 W < \frac{\pi}{2}$$

$$\pi < \lambda_2 W < \frac{3\pi}{2}$$

$$2\pi < \lambda_3 W < \frac{5\pi}{2}$$

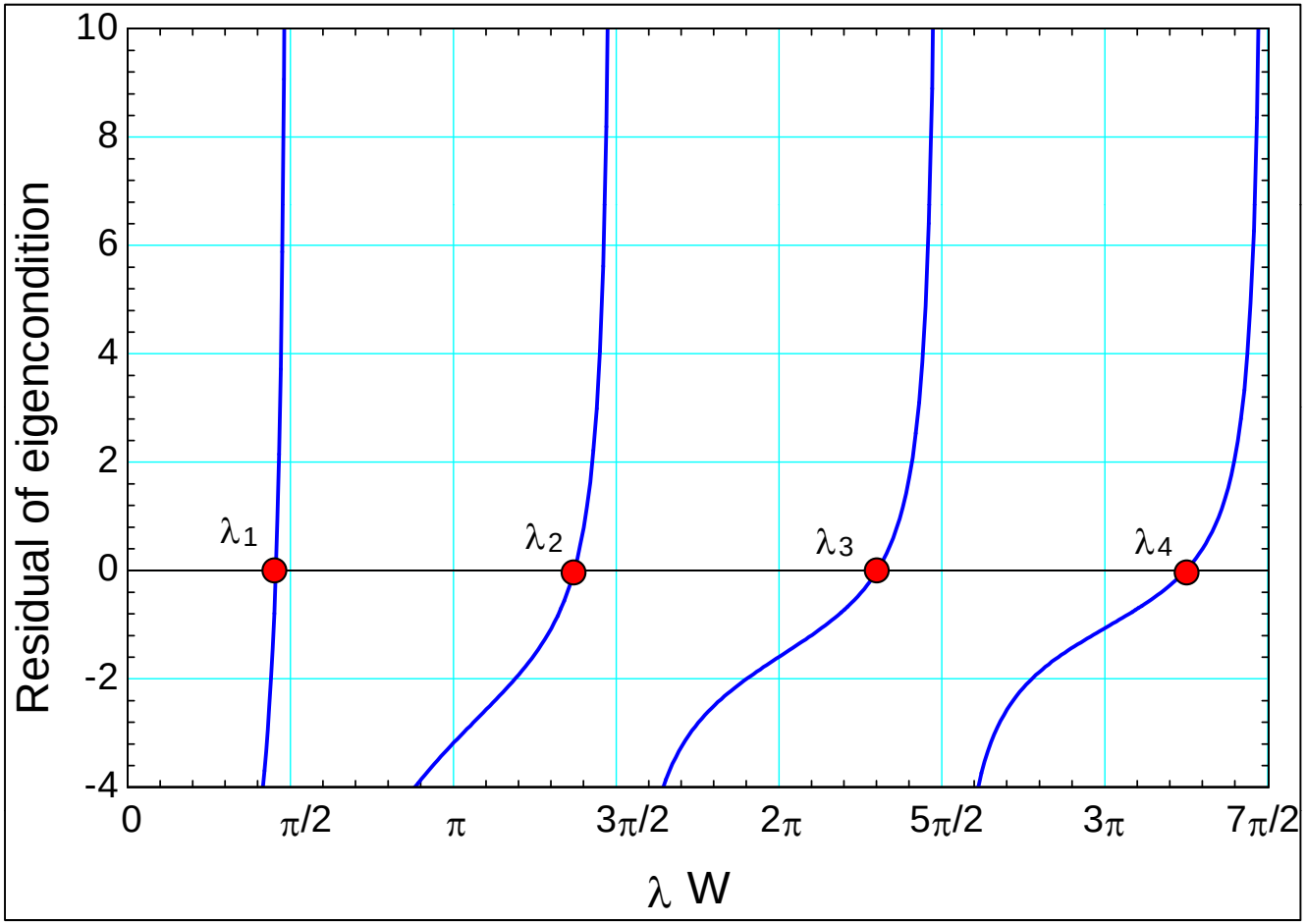
...

in general:

$$(i-1)\pi < \lambda_i W < (i-1)\pi + \frac{\pi}{2}$$

Eigenvalue Ranges from the Residual

- The eigenvalue ranges can also be identified by examining the residual of the eigencondition: $\text{residual} = \tan(\lambda W) - \frac{\bar{h} W}{k(\lambda W)}$



Using EES to Identify Eigenvalues

- Program the eigencondition for each eigenvalue:

```
N=11 [-]                                          "number of terms"
duplicate i=1,N
    tan(lambdaW[i])=h_bar*W/(k*lambdaW[i]) "eigencondition"
    lambda[i]=lambdaW[i]/W                        "eigenvalue"
end
```

- EES identifies the same eigenvalue (λ_1) over and over again...

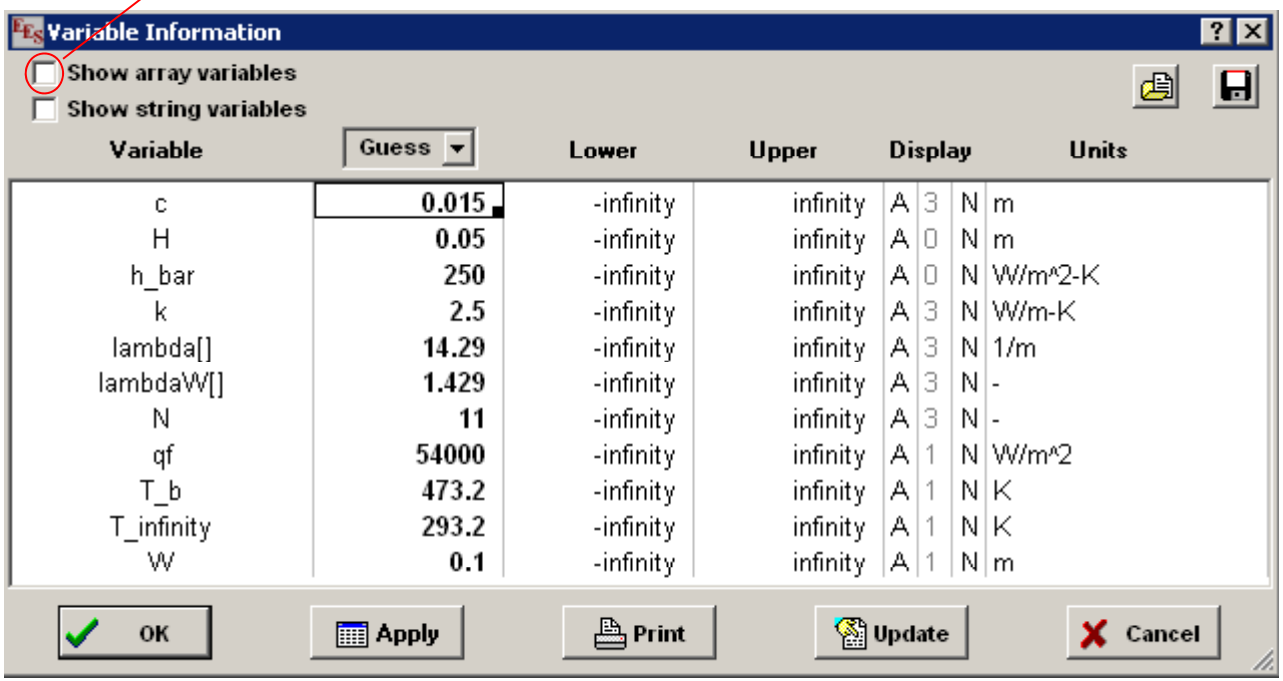
- why?

Sort	1 λ_i [1/m]	2 lambdaW$_i$ [-]
[1]	14.29	1.429
[2]	14.29	1.429
[3]	14.29	1.429
[4]	14.29	1.429
[5]	14.29	1.429
[6]	14.29	1.429
[7]	14.29	1.429
[8]	14.29	1.429
[9]	14.29	1.429
[10]	14.29	1.429
[11]	14.29	1.429

Guess Values and Limits

- The guess value and range of each eigenvalue are not specified
  - select Variable Information from the Options menu
  - note that each variable has a guess and an upper and lower limit associated with it
  - if these quantities are not specified, then EES usually finds the value that is closest to the guess value

de-select Show array variables to collapse arrays



Variable	Guess	Lower	Upper	Display			Units
c	0.015	-infinity	infinity	A	3	N	m
H	0.05	-infinity	infinity	A	0	N	m
h_bar	250	-infinity	infinity	A	0	N	W/m^2-K
k	2.5	-infinity	infinity	A	3	N	W/m-K
lambda[]	14.29	-infinity	infinity	A	3	N	1/m
lambdaW[]	1.429	-infinity	infinity	A	3	N	-
N	11	-infinity	infinity	A	3	N	-
qf	54000	-infinity	infinity	A	1	N	W/m^2
T_b	473.2	-infinity	infinity	A	1	N	K
T_infinity	293.2	-infinity	infinity	A	1	N	K
W	0.1	-infinity	infinity	A	1	N	m

Define Guess Values and Limits

- Setup arrays that contain appropriate limits and guess values for each eigenvalue:

range of each eigenvalue:

$$\underbrace{(i-1)\pi}_{\text{lowerlimit}_i} < \lambda_i W < \underbrace{(i-1)\pi + \frac{\pi}{2}}_{\text{upperlimit}_i}$$

guess value for each eigenvalue:

$$\text{guess}_i = \frac{(\text{lowerlimit}_i + \text{upperlimit}_i)}{2}$$

```
duplicate i=1,N
    lowerlimit[i]=(i-1)*pi                        "lower limit"
    upperlimit[i]=(i-1)*pi+pi/2                   "upper limit"
    guess[i]=(lowerlimit[i]+upperlimit[i])/2      "guess"
end
```

Set Guess Values and Limits

- Set the guess values and limits in the Variable Information window using the arrays:

Variable Information

☐ Show array variables
☐ Show string variables

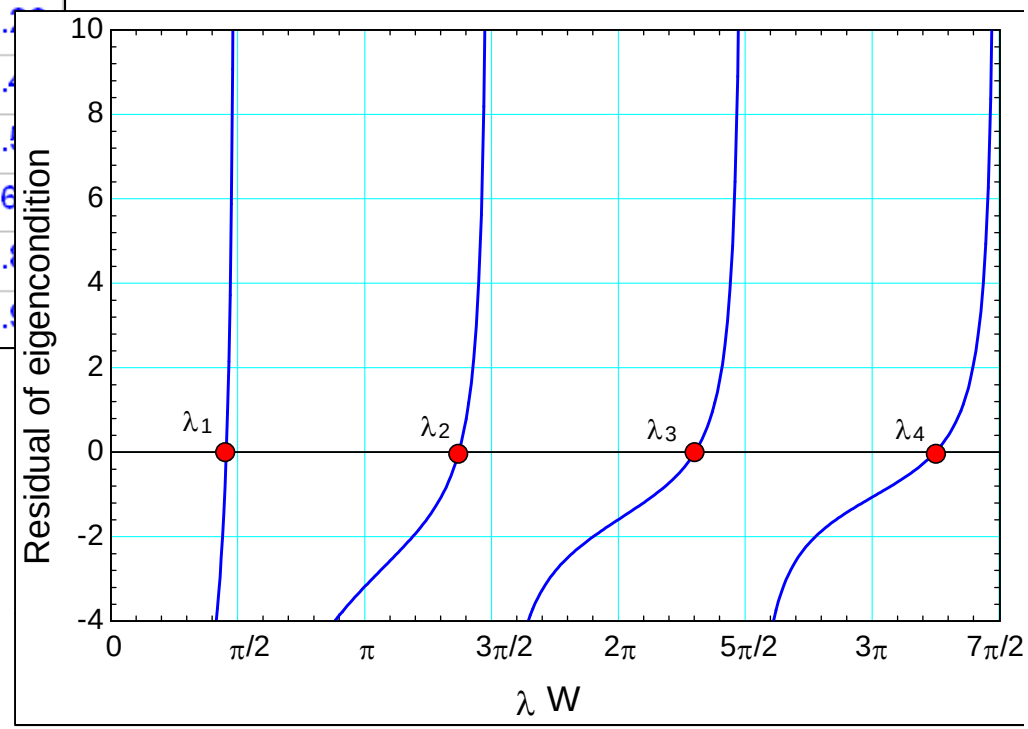
Variable	Guess	Lower	Upper	Display			Units	Key	Comment
c	0.015	-infinity	infinity	A	3	N	m		
guess[]	0.7854	-infinity	infinity	A	3	N	-		
H	0.05	-infinity	infinity	A	0	N	m		
h_bar	250	-infinity	infinity	A	0	N	W/m^2-K		
k	2.5	-infinity	infinity	A	3	N	W/m-K		
lambda[]	14.29	-infinity	infinity	A	3	N	1/m		
lambdaW[]	guess[]	lowerlimit[]	upperlimit[]	A	3	N	-		
lowerlimit[]	0	-infinity	infinity	A	3	N	-		
N	11	-infinity	infinity	A	3	N	-		
qf	54000	-infinity	infinity	A	1	N	W/m^2		
T_b	473.2	-infinity	infinity	A	1	N	K		

OK | Apply | Print | Update | Cancel

Identify Eigenvalues

Sort	1 λ_i [1/m]	2 lambdaW$_i$ [-]	3 guess$_i$ [-]	4 lowerlimit$_i$ [-]	5 upperlimit$_i$ [-]
[1]	14.29	1.429	0.7854	0	1.571
[2]	43.06	4.306	3.927	3.142	4.712
[3]	72.28	7.228	7.069	6.283	7.854
[4]	102	10.2	10.21	9.425	11
[5]	132.1	13.21	13.35	12.57	14.14
[6]	162.6	16.26	16.49	15.71	17.
[7]	193.3	19.33	19.63	18.85	20.
[8]	224.1	22.41	22.78	21.99	23.
[9]	255.1	25.51	25.92	25.13	26
[10]	286.1	28.61	29.06	28.27	29.
[11]	317.2	31.72	32.2	31.42	32.

eigenvalues are correctly defined within each successive range



Solve Non-Homogeneous ODE

- The ODE in the *y*-direction (for each eigenvalue) is:

$$\frac{d^2\theta Y_i}{dy^2} - \lambda_i^2 \theta Y_i = 0 \qquad \theta Y_i = C_{3,i}\cosh(\lambda_i y) + C_{4,i}\sinh(\lambda_i y)$$

- Determine the solution for each eigenvalue:

$$\theta_i = \theta X_i \theta Y_i = C_{1,i}\cos(\lambda_i x)\left[C_{3,i}\cosh(\lambda_i y) + C_{4,i}\sinh(\lambda_i y)\right]$$

- Consolidate the constants:

$$\theta_i = \theta X_i \theta Y_i = \cos(\lambda_i x)\left[C_{3,i}\cosh(\lambda_i y) + C_{4,i}\sinh(\lambda_i y)\right]$$

- Assemble the series solution:

$$\theta = \sum_{i=1}^{N} \theta_i$$

$$\boxed{\theta = \sum_{i=1}^{N} \cos(\lambda_i x)\left[C_{3,i}\cosh(\lambda_i y) + C_{4,i}\sinh(\lambda_i y)\right]}$$

Enforce BC in Non-Homogeneous Direction

- Boundary condition at y = 0: $\theta_{y=0} = T_b - T_\infty$

$$\theta_{y=0} = \sum_{i=1}^{N} \cos(\lambda_i x)\left[C_{3,i}\underbrace{\cosh(\lambda_i 0)}_{1} + C_{4,i}\underbrace{\sinh(\lambda_i 0)}_{0}\right] = T_b - T_\infty$$

$$\sum_{i=1}^{N} C_{3,i}\cos(\lambda_i x) = T_b - T_\infty$$

- Use the orthogonality of the eigenfunctions:
  - multiply by an eigenfunction

$$\sum_{i=1}^{N} C_{3,i}\cos(\lambda_i x)\cos(\lambda_j x) = (T_b - T_\infty)\cos(\lambda_j x)$$

  - and integrate from x = 0 to x = W

$$\sum_{i=1}^{N} C_{3,i}\underbrace{\int_0^W \cos(\lambda_i x)\cos(\lambda_j x)dx}_{=0 \text{ if } i \neq j} = \int_0^W (T_b - T_\infty)\cos(\lambda_j x)dx$$

Enforce BCs in Non-homogeneous Direction

- Equation for $C_{3,i}$:

$$C_{3,i}\underbrace{\boxed{\int_0^W \cos^2\left(\lambda_i x\right)dx}}_{\text{Integral } 1_i} = \underbrace{\int_0^W \left(T_b - T_\infty\right)\cos\left(\lambda_i x\right)dx}_{\text{Integral } 2_i}$$

```
> int((cos(lambda[i]*x))^2,x=0..W);
```

$$\boxed{\frac{1}{2}\frac{\cos(\lambda_i W)\sin(\lambda_i W)+\lambda_i W}{\lambda_i}}$$

```
> int((T_b-T_infinity)*cos(lambda[i]*x),x=0..W);
```

$$-\frac{(-T\_b+T\_infinity)\sin(\lambda_i W)}{\lambda_i}$$

The integrals can conveniently be accomplished in Maple and copied to EES for implementation

```
duplicate i=1,N
    Integral1[i]=1/2*(cos(lambda[i]*W)*sin(lambda[i]*W)+lambda[i]*W)/lambda[i] "copied from Maple"
    Integral2[i]= -(-T_b+T_infinity)*sin(lambda[i]*W)/lambda[i]                "copied from Maple"
    C3[i]*Integral1[i]=Integral2[i]                "constant 3"
end
```

Enforce BCs in Non-Homogeneous Direction

- Boundary condition at y = H:

$$k\left(\frac{\partial\theta}{\partial y}\right)_{y=H} = \begin{cases} \dot{q}'' & \text{for } 0 < x < c \\ 0 & \text{for } c < x < W \end{cases}$$

$$k\sum_{i=1}^{N}\cos(\lambda_i x)\left[\lambda_i C_{3,i}\cosh(\lambda_i H) + \lambda_i C_{4,i}\sinh(\lambda_i H)\right] = \begin{cases} \dot{q}'' & \text{for } 0 < x < c \\ 0 & \text{for } c < x < W \end{cases}$$

- Use the orthogonality of the eigenfunctions (again):
  - multiply by an eigenfunction

$$k\sum_{i=1}^{N}\cos(\lambda_i x)\cos(\lambda_j x)\left[\lambda_i C_{3,i}\cosh(\lambda_i H) + \lambda_i C_{4,i}\sinh(\lambda_i H)\right]$$

$$= \begin{cases} \dot{q}''\cos(\lambda_j x) & \text{for } 0 < x < c \\ 0\cos(\lambda_j x) & \text{for } c < x < W \end{cases}$$

Enforce BCs in Non-Homogeneous Direction

- Use orthogonality of eigenfunctions (continued):
  - integrate from x = 0 to x = W

$$k\sum_{i=1}^{N}\left[\lambda_i C_{3,i}\cosh\left(\lambda_i H\right)+\lambda_i C_{4,i}\sinh\left(\lambda_i H\right)\right]\underbrace{\int_0^W \cos\left(\lambda_i x\right)\cos\left(\lambda_j x\right)dx}_{=0 \text{ if } i\neq j}$$

$$=\int_0^c \dot{q}''\cos\left(\lambda_j x\right)dx+\int_c^W 0\cos\left(\lambda_j x\right)dx$$

- Leads to an expression for each constant C4,i:

$$k\left[\lambda_i C_{3,i}\cosh\left(\lambda_i H\right)+\lambda_i C_{4,i}\sinh\left(\lambda_i H\right)\right]\underbrace{\int_0^W \cos^2\left(\lambda_j x\right)dx}_{\text{Integral1}_i}=\underbrace{\int_0^c \dot{q}''\cos\left(\lambda_j x\right)dx}_{\text{Integral3}_i}$$

Enforce BCs in Non-Homogeneous Direction

$$k\left[\lambda_i C_{3,i}\cosh\left(\lambda_i H\right)+\lambda_i C_{4,i}\sinh\left(\lambda_i H\right)\right]\underbrace{\int_0^W \cos^2\left(\lambda_j x\right)dx}_{\text{Integral1}_i}=\underbrace{\int_0^c \dot{q}''\cos\left(\lambda_j x\right)dx}_{\text{Integral3}_i}$$

```
> int(qf*cos(lambda[i]*x),x=0..c);
```

$$\frac{qf\sin(\lambda_i c)}{\lambda_i}$$

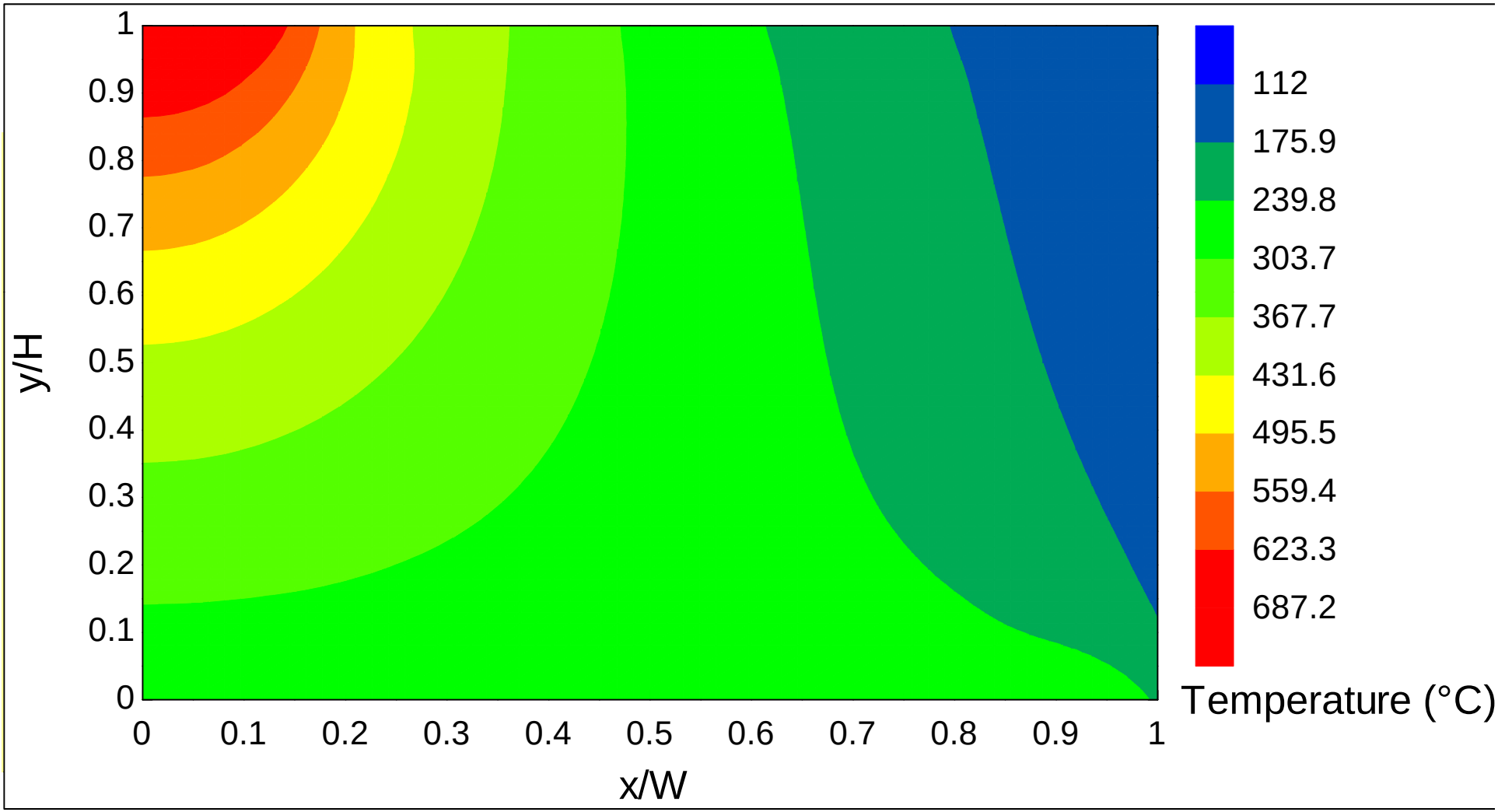
```
duplicate i=1,N
    Integral3[i]=qf/lambda[i]*sin(lambda[i]*c) "copied from Maple"
    k*(lambda[i]*C3[i]*cosh(lambda[i]*H)+lambda[i]*C4[i]*sinh(lambda[i]*H))*Integral1[i]=Integral3[i]
end
```

Implement the Solution

$$\theta = \sum_{i=1}^{N} \cos(\lambda_i x)\left[C_{3,i}\cosh(\lambda_i y) + C_{4,i}\sinh(\lambda_i y)\right]$$

```
x_bar=0.5 [-]                                   "dimensionless x-position"
y_bar=0.5 [-]                                   "dimensionless y-position"
x=x_bar*W                                       "x-position"
y=y_bar*H                                       "y-position"
duplicate i=1,N
    theta[i]=cos(lambda[i]*x)*(C3[i]*cosh(lambda[i]*y)+C4[i]*sinh(lambda[i]*y))
                                                "i'th term in the series solution"
end
theta=sum(theta[i],i=1,N)                       "series solution"
T=theta+T_infinity                              "temperature"
T_C=converttemp(K,C,T)                          "in C"
```

Temperature Distribution



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Module 6

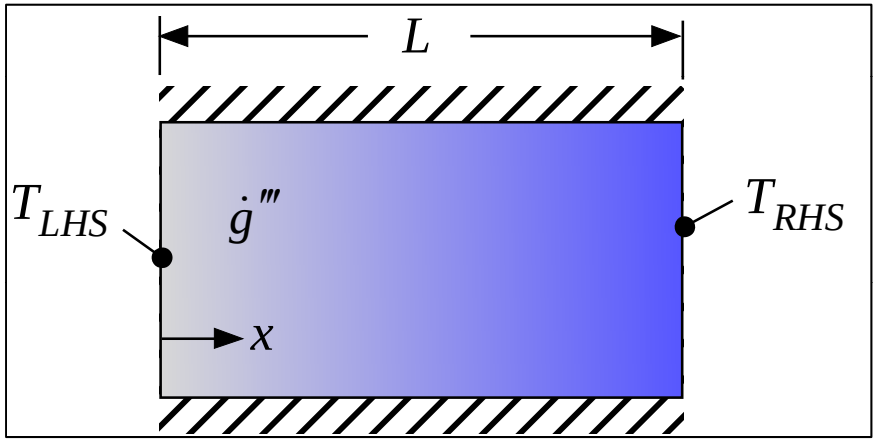
Superposition

1. G. Nellis and S. A. Klein (2009), Heat Transfer, Cambridge University Press Lecture notes provided by Drs. Nellis and Klein

2. Other textbooks such as T.L. Bergman, A.S. Lavine, F.P. Incropera, D.P. DeWitt, "Fundamentals of Heat and Mass Transfer", 8th Edition

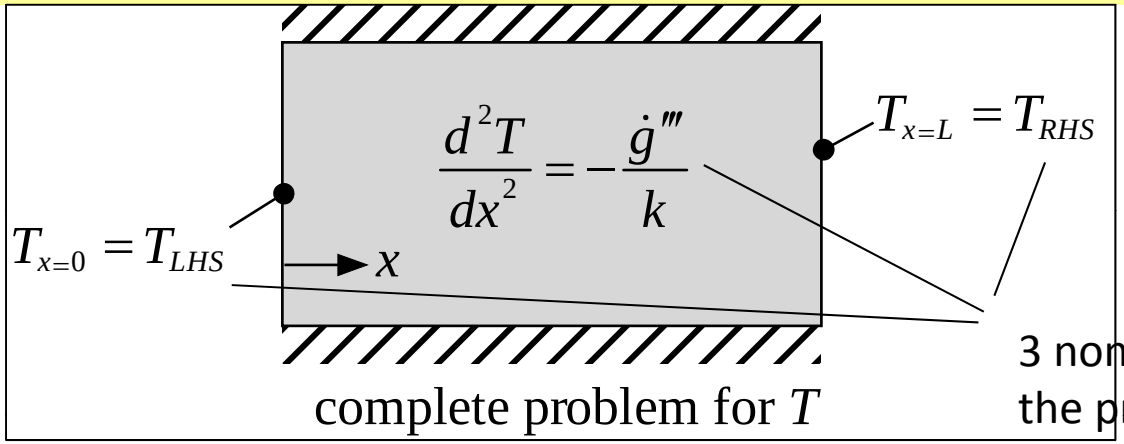
Superposition

- Superposition allows you to split a problem with many non-homogeneous components (e.g., BCs or terms in the PDE) into sub-problems that each contain only a single non-homogeneous component
- For example:
  - a 1-D plane wall with a uniform rate of volumetric generation



  - This problem has three non-homogeneous components:
    - the generation term in the PDE
    - the non-homogeneous BC at x = 0
    - the non-homogeneous BC at x = L

Superposition Applied to a 1-D Problem

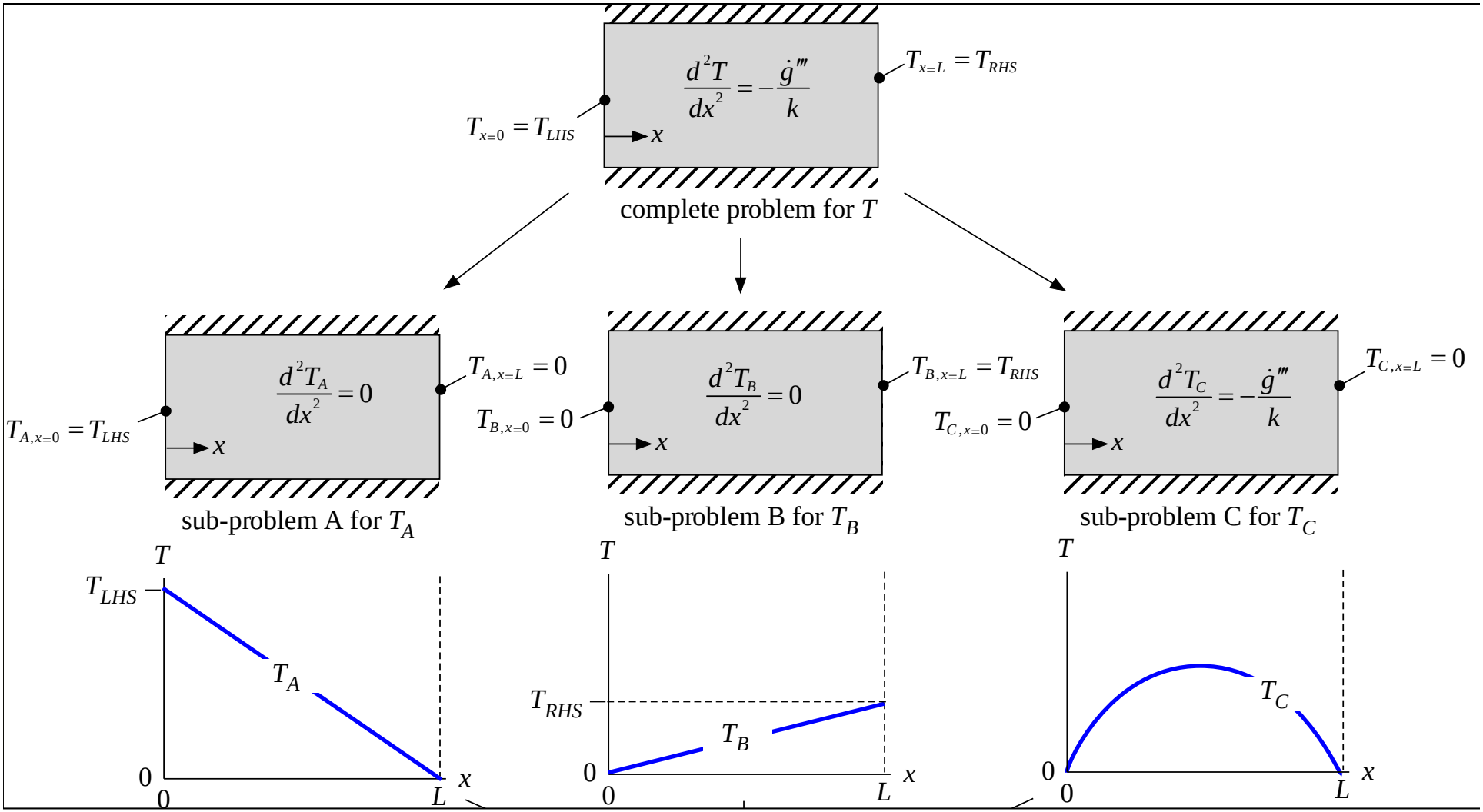


3 non-homogeneous components of the problem

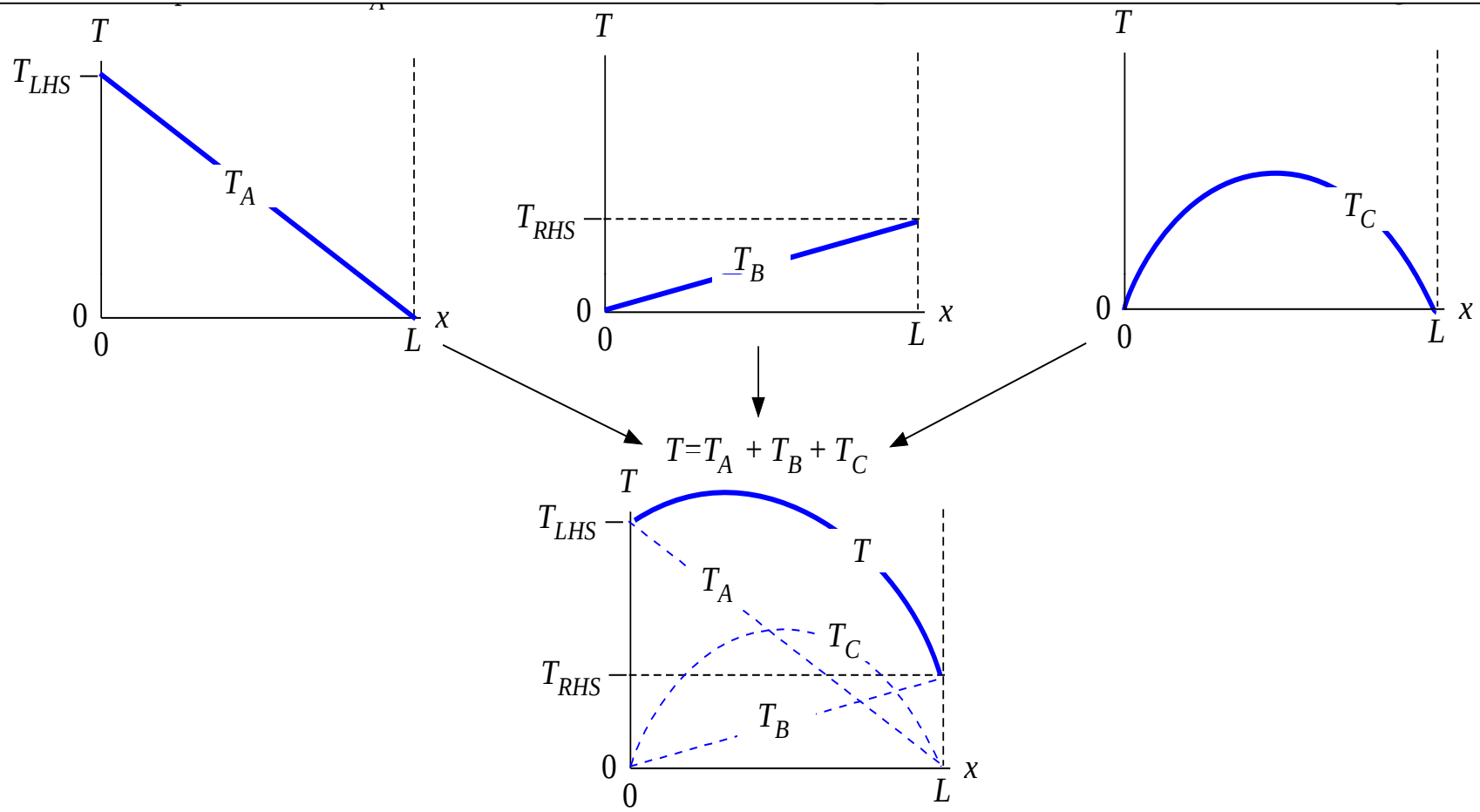
- This problem can be split into three sub-problems that each contain only one of the non-homogeneous components:
  - sub-problem A considers the LHS BC
    - RHS BC and PDE are replaced with their homogeneous equivalents
  - sub-problem B considers the RHS BC
    - LHS BC and PDE are replaced with their homogeneous equivalents
  - sub-problem C considers the generation term in the PDE
    - BCs are replaced with their homogeneous equivalents
- Superposition allows the solution to be written as:

$$T = T_A + T_B + T_C$$

Break Problem into Sub-Problems and Solve



Add Solutions of the Sub-Problems

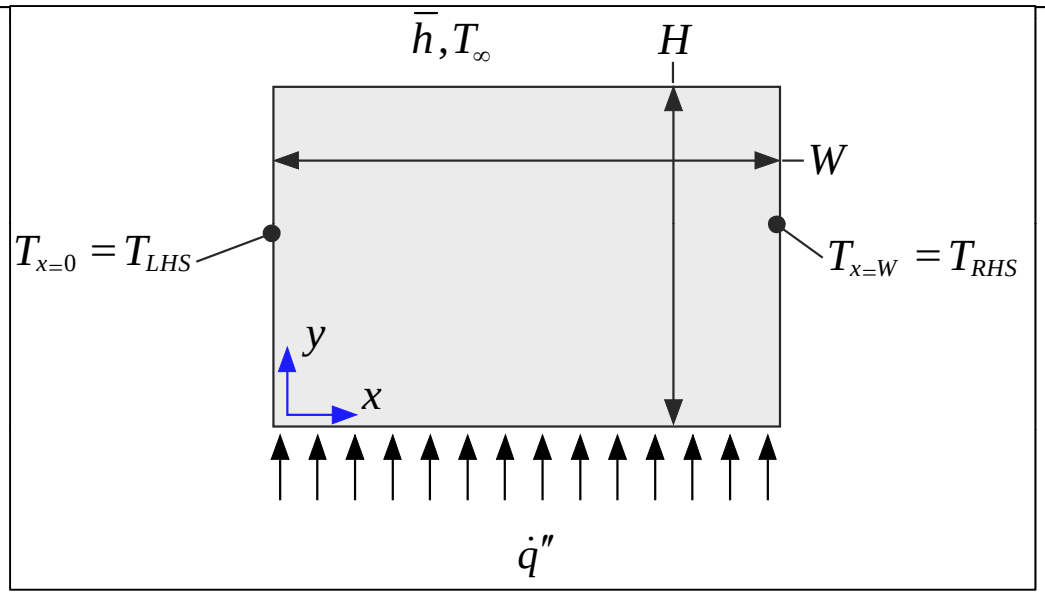


Superposition for Separation of Variables

- Real advantage of superposition apparent when applied to 2-D or 3-D problems
- Separation of variables requires homogeneous BC in one direction
  - Real problems rarely satisfy this criterion
  - Simple transformations not usually sufficient
- Superposition allows subdivision into two sub-problems
  - Sub-problem A replaces non-homogeneous BCs in x-direction with homogeneous equivalents
  - Sub-problem B replaces non-homogeneous BCs in y-direction with homogeneous equivalents
  - These sub-problems can each be solved using separation of variables
  - Solution is sum (superposition) of these solutions to the sub-problems

Problem Specification

- A problem with all non-homogeneous BCs
  - there is no homogeneous direction for this problem
  - it cannot be solved using separation of variables



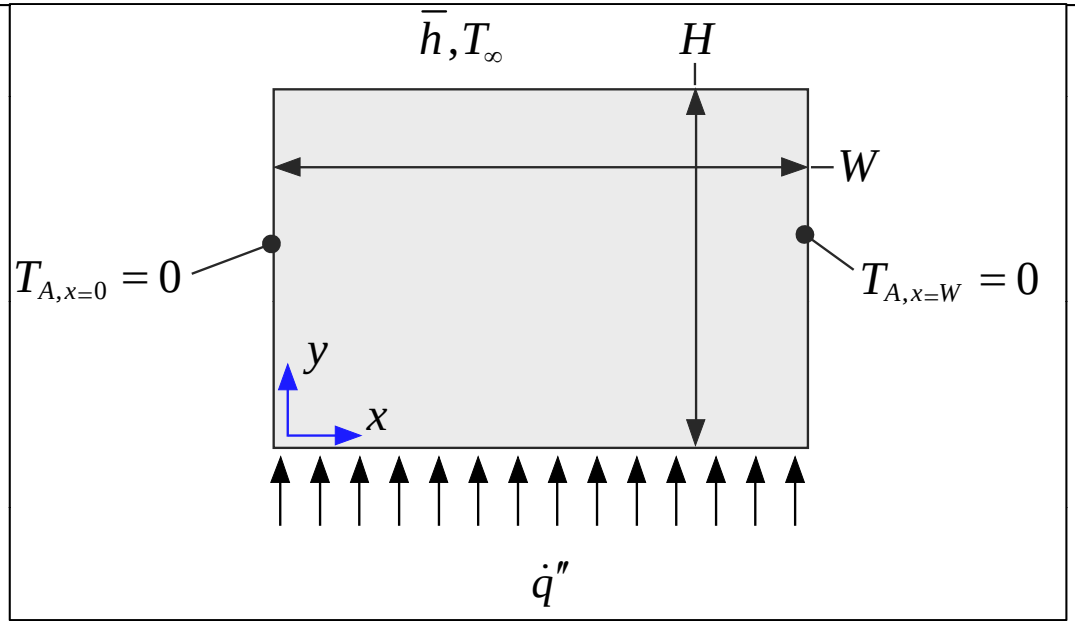
partial differential equation: $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0$

x-direction boundary conditions: $T_{x=0} = T_{LHS} \quad T_{x=W} = T_{RHS}$

y-direction boundary conditions: $\dot{q}'' = -k\left(\dfrac{\partial T}{\partial y}\right)_{y=0} \quad -k\left(\dfrac{\partial T}{\partial y}\right)_{y=H} = \bar{h}\left(T_{y=H} - T_\infty\right)$

Sub-Problem A

- Sub-problem A replaces the BCs in the x-direction with their homogeneous equivalent
  - sub-problem A can be solved using separation of variables (x is the homogeneous direction)



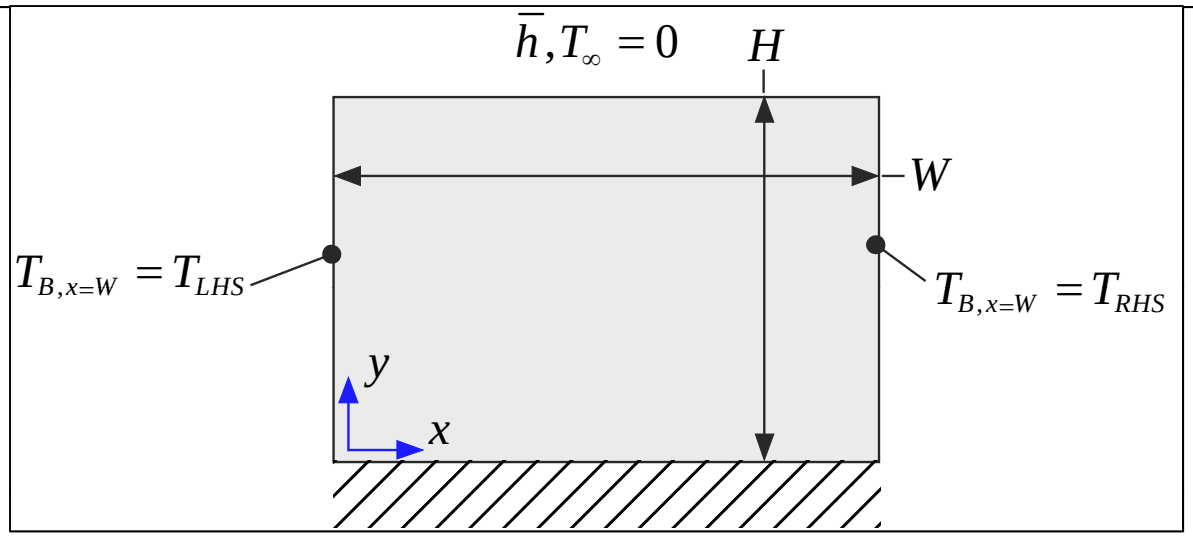
partial differential equation: $\dfrac{\partial^2 T_A}{\partial x^2} + \dfrac{\partial^2 T_A}{\partial y^2} = 0$

x-direction boundary conditions: $T_{A,x=0} = 0 \quad T_{A,x=W} = 0$

y-direction boundary conditions: $\dot{q}'' = -k\left(\dfrac{\partial T_A}{\partial y}\right)_{y=0} \quad -k\left(\dfrac{\partial T_A}{\partial y}\right)_{y=H} = \bar{h}\left(T_{A,y=H} - T_\infty\right)$

Sub-Problem B

- Sub-problem B replaces the BCs in the y-direction with their homogeneous equivalent
  - sub-problem B can also be solved using separation of variables (y is the homogeneous direction)



partial differential equation: $\dfrac{\partial^2 T_B}{\partial x^2} + \dfrac{\partial^2 T_B}{\partial y^2} = 0$

x-direction boundary conditions: $T_{B,x=0} = T_{LHS} \quad T_{B,x=W} = T_{RHS}$

y-direction boundary conditions: $-k\left(\dfrac{\partial T_B}{\partial y}\right)_{y=0} = 0 \quad -k\left(\dfrac{\partial T_B}{\partial y}\right)_{y=H} = \bar{h}\, T_{B,y=H}$

Superposition

- The sum of the solutions to sub-problems A and B satisfies the original PDE and all of the original BCs

$$T = T_A + T_B$$

PDE: $$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \rightarrow \frac{\partial^2 \left(T_A + T_B\right)}{\partial x^2} + \frac{\partial^2 \left(T_A + T_B\right)}{\partial y^2} = 0$$

$$\underbrace{\frac{\partial^2 T_A}{\partial x^2} + \frac{\partial^2 T_A}{\partial y^2}}_{=0 \text{ for sub-problem A}} + \underbrace{\frac{\partial^2 T_B}{\partial x^2} + \frac{\partial^2 T_B}{\partial y^2}}_{=0 \text{ for sub-problem B}} = 0$$

x-direction BCs: $$T_{x=0} = T_{LHS} \rightarrow \underbrace{T_{A,x=0}}_{\substack{=0 \text{ for} \\ \text{sub-problem A}}} + \underbrace{T_{B,x=0} = T_{LHS}}_{\text{sub-problem B BC}}$$

$$T_{x=W} = T_{RHS} \rightarrow \underbrace{T_{A,x=W}}_{\substack{=0 \text{ for} \\ \text{sub-problem A}}} + \underbrace{T_{B,x=W} = T_{RHS}}_{\text{sub-problem B BC}}$$

Superposition

- The sum of the solutions to sub-problems A and B satisfies the original PDE and all of the original BCs

$$T = T_A + T_B$$

y-direction BCs: $$-k\left(\frac{\partial T}{\partial y}\right)_{y=0} = \dot{q}'' \rightarrow \underbrace{-k\left(\frac{\partial T_B}{\partial y}\right)_{y=0}}_{=0 \text{ for sub-problem B}} \underbrace{-k\left(\frac{\partial T_A}{\partial y}\right)_{y=0} = \dot{q}''}_{\text{BC for sub-problem A}}$$

$$-k\left(\frac{\partial T}{\partial y}\right)_{y=H} = \bar{h}\left(T_{y=H} - T_\infty\right) \rightarrow \overbrace{-k\left(\frac{\partial T_B}{\partial y}\right)_{y=H}}^{\text{LHS of BC for sub-problem B}} \underbrace{-k\left(\frac{\partial T_A}{\partial y}\right)_{y=H}}_{\text{LHS of BC for sub-problem A}} = \overbrace{\bar{h}\,T_{B,y=H}}^{\text{RHS of BC for sub-problem B}} + \underbrace{\bar{h}\left(T_{A,y=H} - T_\infty\right)}_{\text{LHS of BC for sub-problem A}}$$

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Module 7
Numerical Methods: 2-D Conduction with Heat Generation

1. G. Nellis and S. A. Klein (2009), Heat Transfer, Cambridge University Press Lecture notes provided by Drs. Nellis and Klein

2. Other textbooks such as T.L. Bergman, A.S. Lavine, F.P. Incropera, D.P. DeWitt, "Fundamentals of Heat and Mass Transfer", 8th Edition

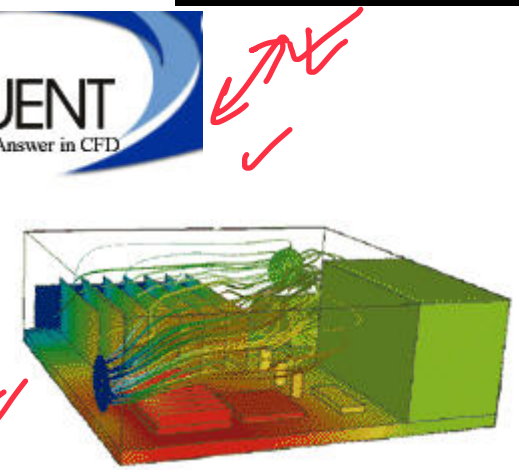
Introduction to Finite Difference Method

Several different numerical methods have been developed.

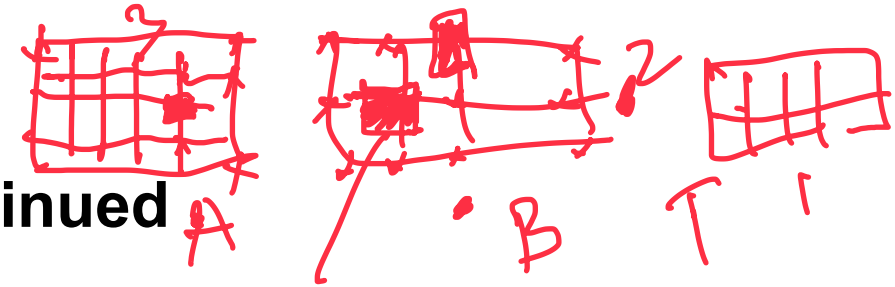
Finite difference method is more intuitive and easy to apply.

Other methods include

- Finite element method
- Finite volume method

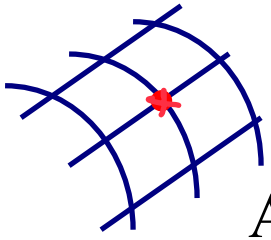


$T(x,y)$

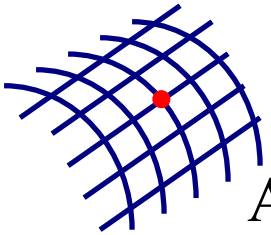


Numerical method -- continued

Discretization: a *nodal network*, called *mesh* or *grid*. We can refine the mesh but we cannot obtain continuous solutions.



A coarse mesh

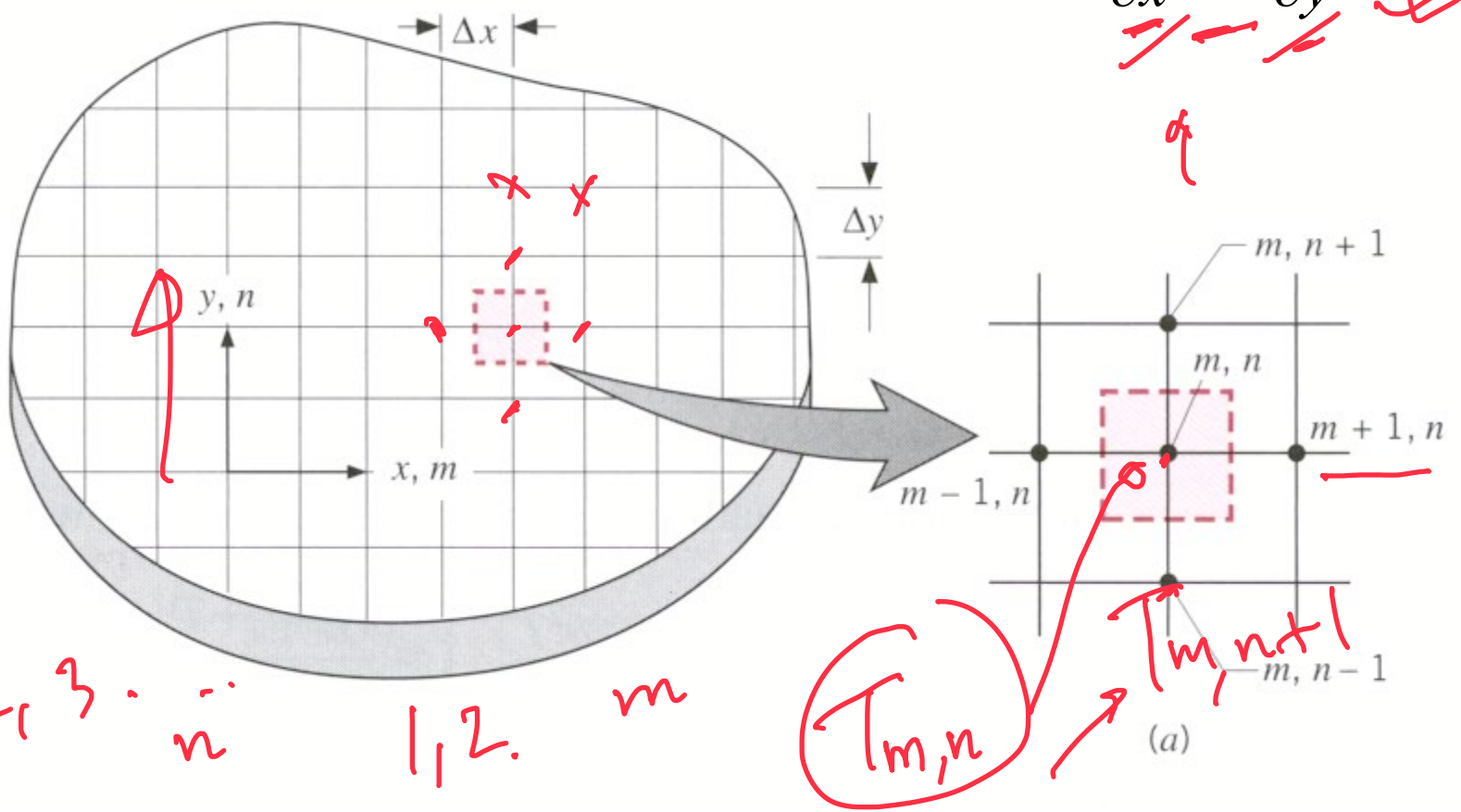


A fine mesh

Nodal points or simply *nodes*, nodal property such as *nodal temperature* - it is the temperature of the node but it also represents the temperature around the node (average).

Basic principle: Through approximation, convert the differential equations to a set of algebraic equations that can be solved numerically.

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$



$T_{m,n}$ = average temperature around node $P(m, n)$.

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{m,n+1} + T_{m,n-1} - 2T_m}{\Delta y^2}$$

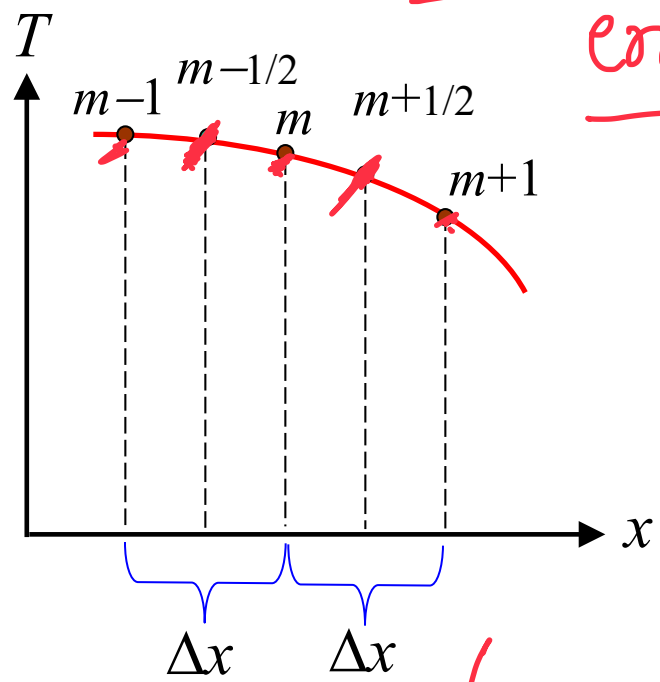
Discretization

Discretization of a Continuous Function

Finite Difference

$$\left.\frac{dT}{dx}\right|_{m+1/2} \approx \frac{T_{m+1}-T_m}{\Delta x}$$

$$\left.\frac{dT}{dx}\right|_{m-1/2} \approx \frac{T_m-T_{m-1}}{\Delta x}$$

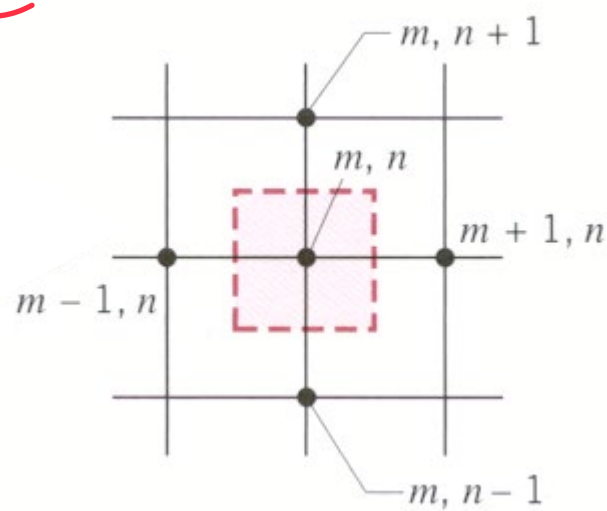


$$\left.\frac{d^2T}{dx^2}\right|_m \approx \frac{\left.\frac{\partial T}{\partial x}\right|_{m+1/2} - \left.\frac{\partial T}{\partial x}\right|_{m-1/2}}{\Delta x} = \frac{T_{m+1}+T_{m-1}-2T_m}{(\Delta x)^2}$$

$$\Rightarrow (T_{m+1} - T_m) - (T_m - T_{m-1}) / \Delta x^2$$

2-D Steady State

$$\frac{k}{k}\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right]=0$$



$$\left.\frac{\partial^2 T}{\partial x^2}\right|_m=\frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_m=\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}$$

If $\Delta x=\Delta y$, then we obtain

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

Finite difference equation for (m, n)

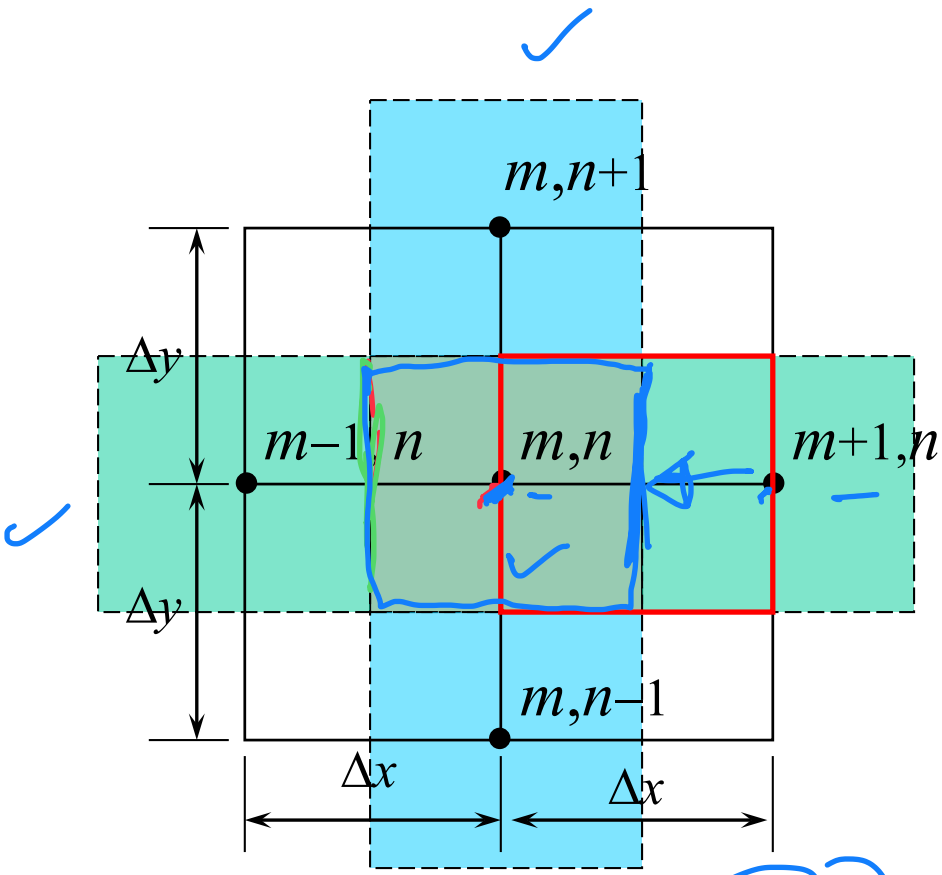
Use N linear algebraic equations to solve N unknowns.

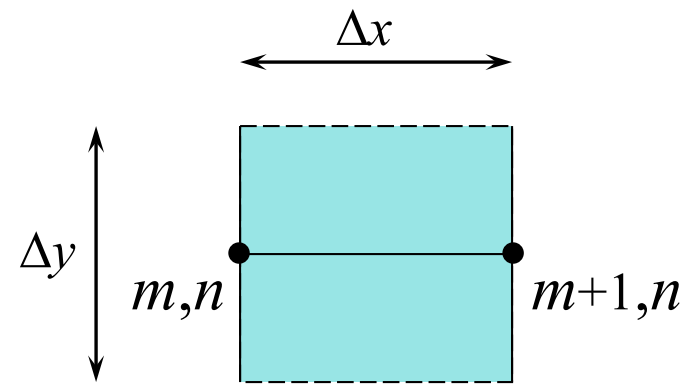
$$\frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{\cancel{\Delta x^2}} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{\cancel{\Delta y^2}} = 0$$

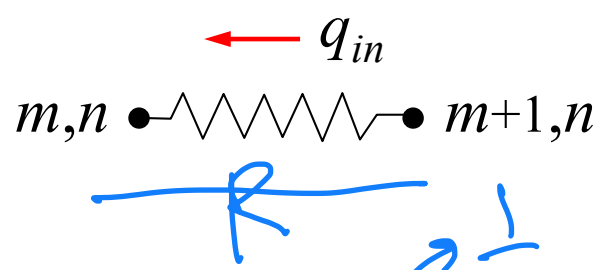
$\Rightarrow$

$$R = \frac{L}{kA} = \frac{\Delta x}{k \Delta y \cdot 1}$$

The Energy Balance Method





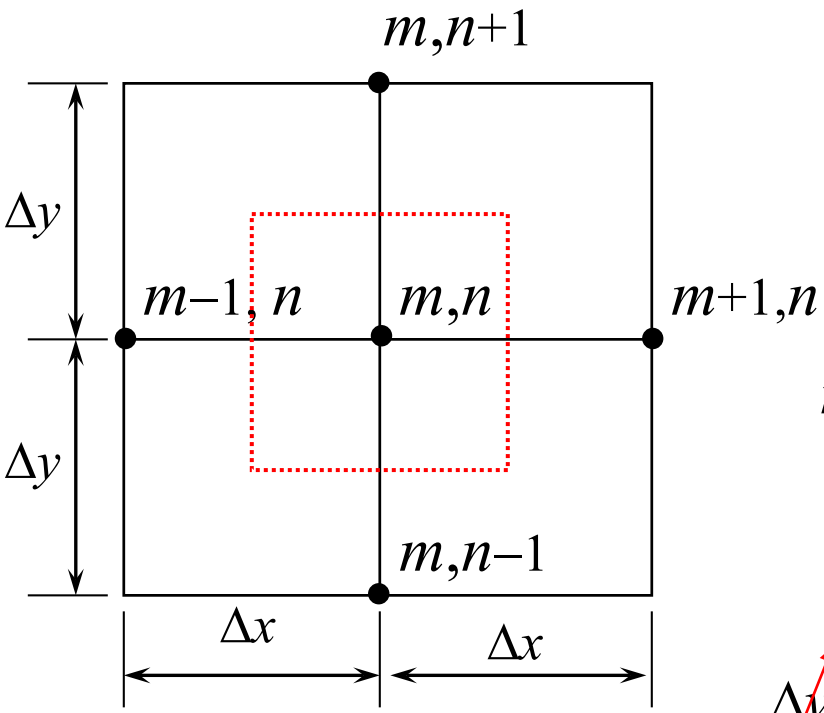


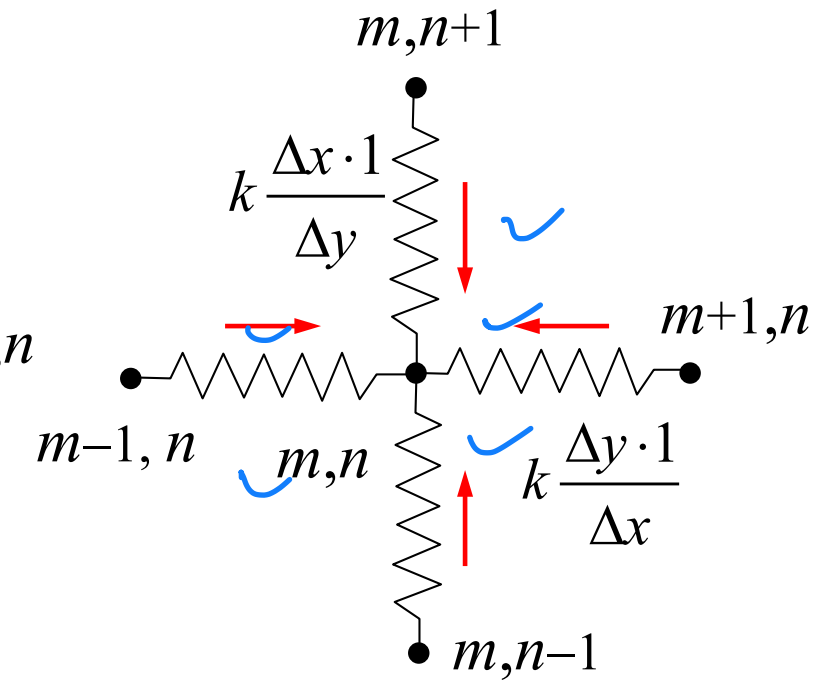
$$\frac{\Delta T}{R}$$

$$q_{in} = \left(k \frac{\Delta y \cdot 1}{\Delta x}\right)(T_{m+1,n} - T_{m,n})$$

$$\frac{1}{R}$$

2-D Steady State with Heat Generation



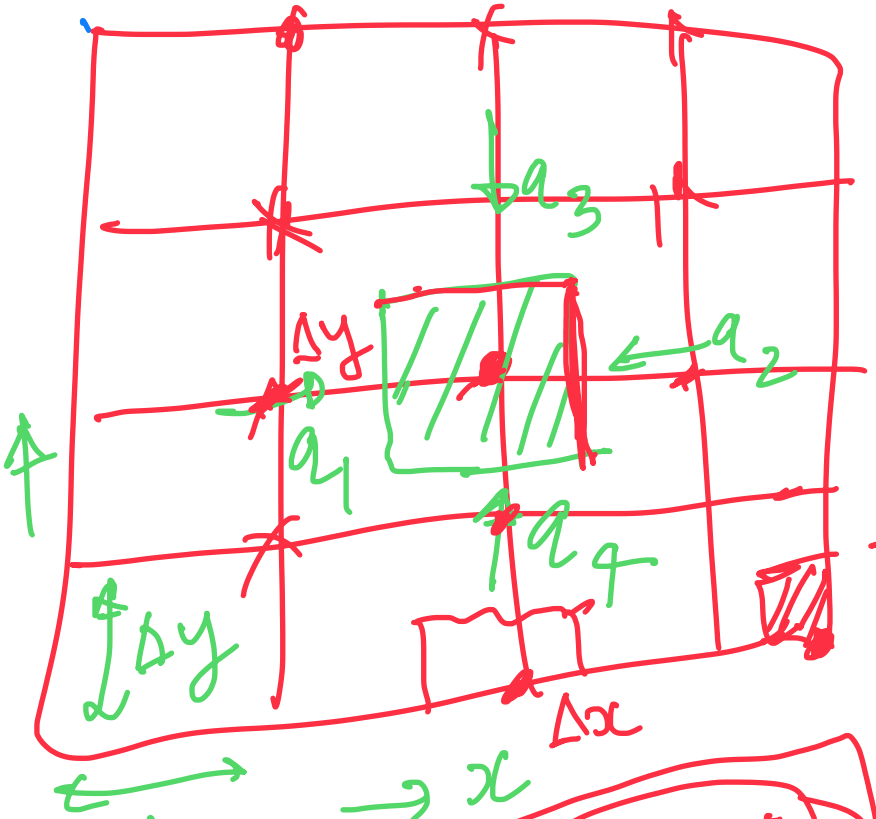


$$\dot{E}_{in} + \dot{E}_g = 0$$

$$k\frac{\Delta y}{\Delta x}(T_{m+1,n} - T_{m,n}) + k\frac{\Delta y}{\Delta x}(T_{m-1,n} - T_{m,n})$$

$$+k\frac{\Delta x}{\Delta y}(T_{m,n+1} - T_{m,n}) + k\frac{\Delta x}{\Delta y}(T_{m,n-1} - T_{m,n})$$

$$+\dot{q}\Delta x\Delta y = 0$$



(W) $\dot{E}_{in} - \cancel{\dot{E}_{out}} + \dot{E}_g = 0$

$$\Rightarrow q_1 + q_2 + q_3 + q_4 + \dot{q}\,\Delta x\,\Delta y\,1 = 0$$

$$\Rightarrow \left[\frac{k_x \Delta y\,(T_{m+1,n} - T_{m,n})}{\Delta x}\right] + \left[\frac{k_x \cdot \Delta y\,(T_{m+1,n} - T_{m,n})}{\Delta x}\right] + \left[\frac{k_y \Delta x\,(T_{m,n+1} - T_{m,n})}{\Delta y}\right] + \left[\frac{k_y \Delta x\,(T_{m,n-1} - T_{m,n})}{\Delta y}\right] + \dot{E}_g = 0$$

k_x & k_y

$$= \frac{kA\,\partial T}{\partial x}$$

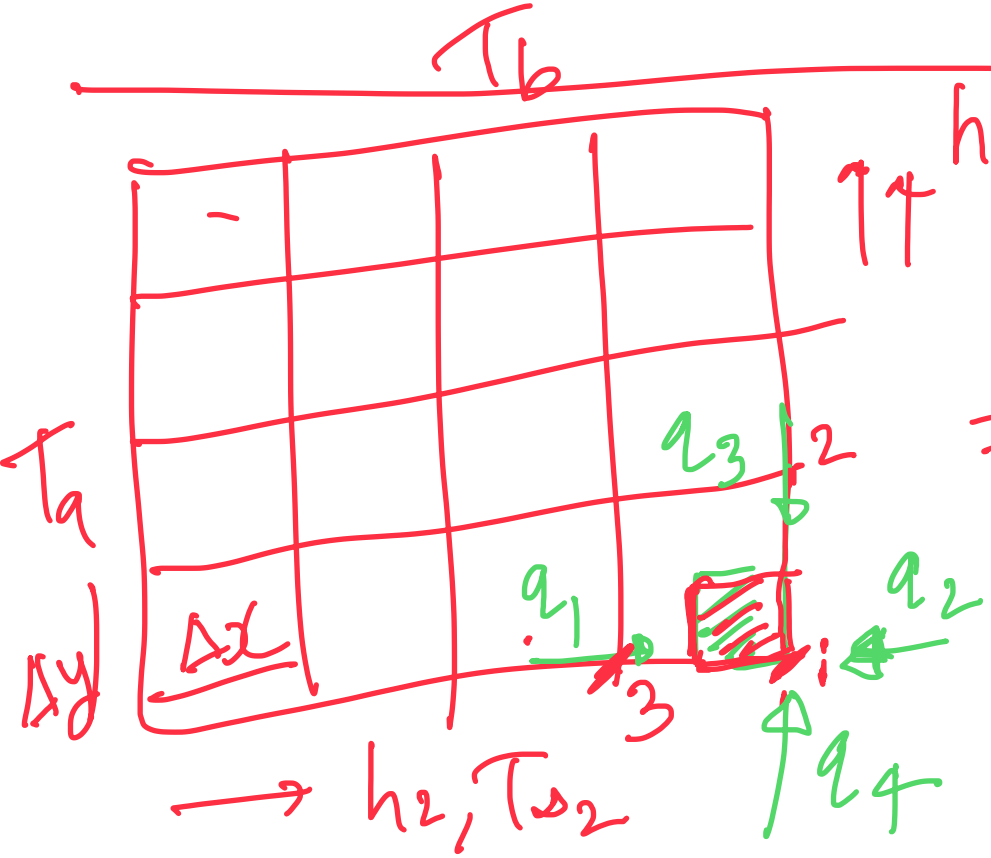
$\dot{q} \equiv W/m^3$

heat generation

$$\frac{\Delta T}{R}$$

Assume, $k_x = k_y$ | $\Delta x = \Delta y$

$$\left[T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n}\right] + \dot{q}\,\Delta x\,\Delta y = 0$$



$$\Delta V = \frac{\Delta x}{2} \cdot \frac{\Delta y}{2} \cdot 1$$

$$\dot{E}_{in} - \cancel{\dot{E}_{out}} + \dot{E}_g = 0$$

$$\Rightarrow q_1 + q_2 + q_3 + q_4 + \dot{E}_g = 0$$

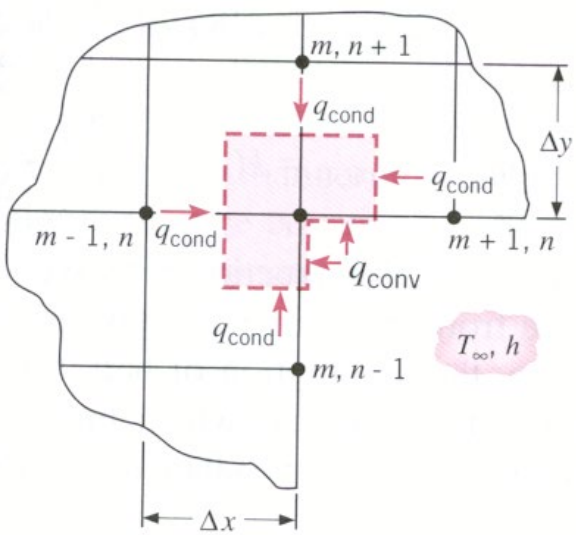
$$\Rightarrow K_x \cdot \frac{\Delta y}{2}\left(\frac{T_3 - T_1}{\Delta x}\right) + h_1 \cdot \frac{\Delta y}{2}\left(T_{\infty 1} - T_1\right)$$

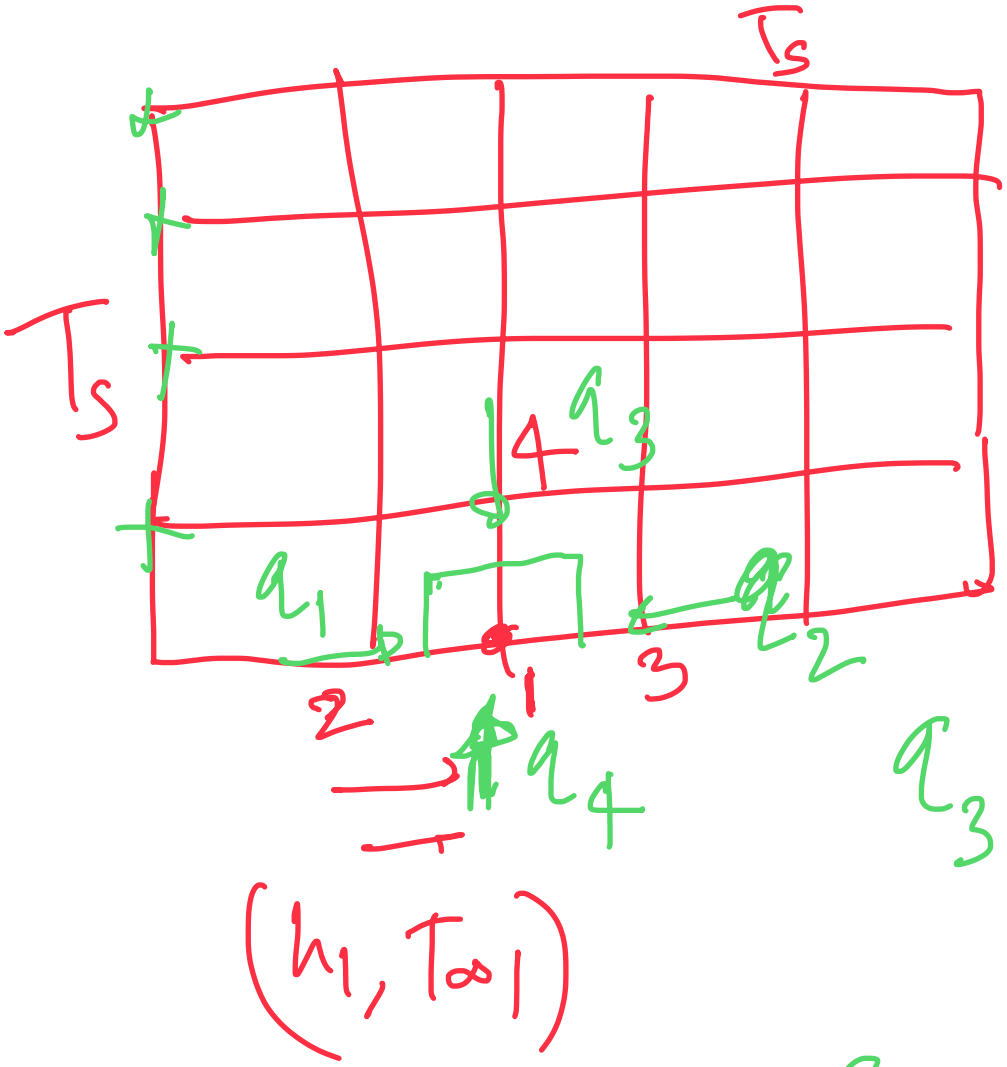
$$+ K_y \frac{\Delta x}{2}\left(\frac{T_2 - T_1}{\Delta y}\right) + h_2 \frac{\Delta x}{2}\left(T_{\infty 2} - T_1\right)$$

$$+ \dot{q} \frac{\Delta x \cdot \Delta y}{4} = 0$$

If $\Delta x = \Delta y$,

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{\dot{q}(\Delta x)^2}{k} = 0$$





$h_2, T_{\infty 2}$

$$\Delta V = \Delta x \cdot \frac{\Delta y}{2}$$

$$q_1 \rightarrow k_x \frac{\Delta y}{2} \frac{(T_2 - T_1)}{\Delta x}$$

$$q_2 \rightarrow k_x \frac{\Delta y}{2} \frac{(T_3 - T_1)}{\Delta x}$$

$$q_3 \rightarrow k_y \Delta x \frac{(T_4 - T_1)}{\Delta y}$$

$$q_4 \rightarrow h_1 \cdot \Delta x (T_{\infty 1} - T_1)$$

$$\left[q_1 + q_2 + q_3 + q_4 + \Delta x \cdot \Delta y / 2 = 0 \right]$$

a_{12} $(T_2) \rightarrow \dfrac{k_x \cdot \Delta y}{2 \cdot \Delta x}$

$C_1 = -h_1 \Delta x T_{\infty 1}$

a_{11} $(T_1) \rightarrow -\dfrac{k_x \Delta y}{2\Delta x} - \dfrac{k_x \cdot \Delta y}{2\Delta x} - \dfrac{k_y \Delta x}{\Delta y}$

$+ h_1 \Delta x T_{\infty 1}$

a_{15} $T_5 \rightarrow 0$

$$a_{11} T_1 + a_{12} T_2 + \cdots \cdots + a_{1n} T_N$$

$$= C_N$$

What do we end up with is a set of linear algebraic equations. Assume there are N unknown nodal temperatures.

$$a_{11}T_1 + a_{12}T_2 + \ldots\ldots\ldots + a_{1N}T_N = C_1$$
$$a_{21}T_1 + a_{22}T_2 + \ldots\ldots\ldots + a_{2N}T_N = C_2$$

$$\ldots\ldots\ldots$$

$$a_{N1}T_1 + a_{N2}T_2 + \ldots\ldots\ldots + a_{NN}T_N = C_N$$

$$\sum_{j=1}^{N} a_{ij}T_j = C_i, \quad i = 1,\ 2, \ldots,\ N$$

$$\overbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2N} \\ & & & & \\ & \cdots & \cdots & \cdots & \\ a_{N1} & a_{N2} & \cdots & \cdots & a_{NN} \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} T_1 \\ T_2 \\ \cdot \\ \cdot \\ T_N \end{bmatrix}}^{T} = \overbrace{\begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ \cdot \\ C_N \end{bmatrix}}^{C}$$

$$[A]_{N\times N}[T]_{N\times 1} = [C]_{N\times 1}$$

Matrix inversion solution:

$$[T] = [A]^{-1}[C]$$

$T_{m,n}$ $\quad m=1,10$ $\quad n=1,5$ $\quad 50$ $\quad T_{50}$

1. Energy balance
2. Grid-Independent Study
3. Compare with exact solution if available

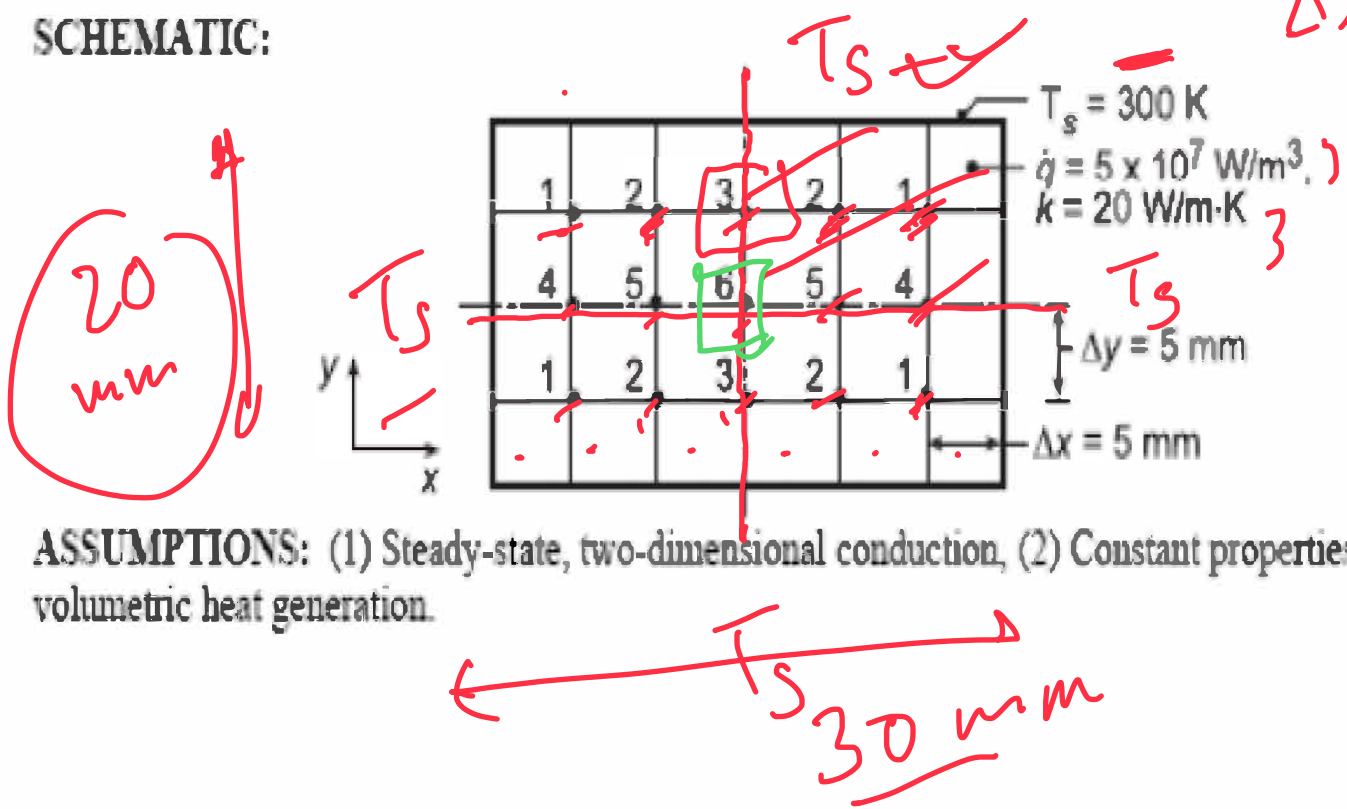
PROBLEM

KNOWN: Volumetric heat generation in a rectangular rod of uniform surface temperature.

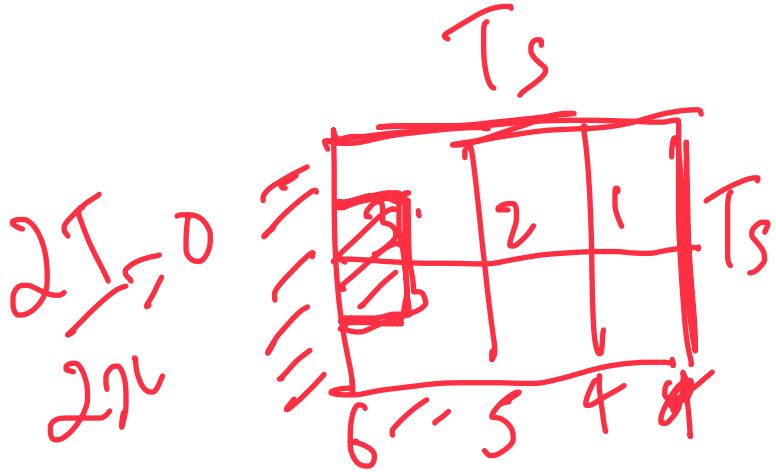
FIND: (a) Temperature distribution in the rod, and (b) With boundary conditions unchanged, heat generation rate causing the midpoint temperature to reach 600 K.

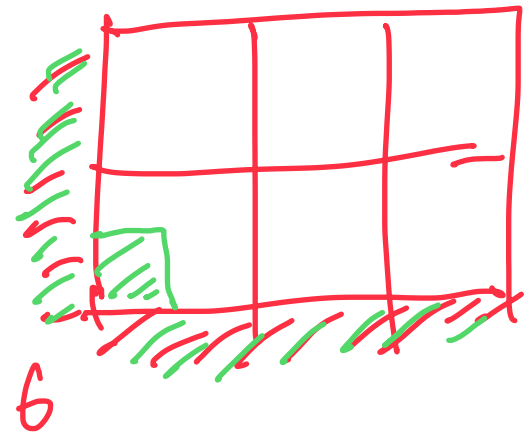
$\Delta x = \Delta y = 5\,mm$

SCHEMATIC:



ASSUMPTIONS: (1) Steady-state, two-dimensional conduction, (2) Constant properties, (3) Uniform volumetric heat generation.





Last problem of $\boxed{\text{Hw. 1}}$

$k = 490 \frac{W}{mK} \rightarrow SiC$

$R_{fin} = \frac{1}{\eta_f h A_f}$

$\eta_f = \frac{\tanh(mL_c)}{(mL_c)}$

Table 1-2: A summary of common resistance formulae.

Situation	Resistance formula	Nomenclature
Plane wall	$R_{pw} = \dfrac{L}{k\,A_c}$	L = wall thickness (|| to heat flow) k = conductivity A_c = cross-sectional area ($\perp$ to heat flow)
Cylinder (radial heat transfer)	$R_{cyl} = \dfrac{\ln\left(\dfrac{r_{out}}{r_{in}}\right)}{2\,\pi\,L\,k}$	L = cylinder length k = conductivity r_{in} and r_{out} = inner and outer radii
Sphere (radial heat transfer)	$R_{sph} = \dfrac{1}{4\,\pi\,k}\left[\dfrac{1}{r_{in}} - \dfrac{1}{r_{out}}\right]$	k = conductivity r_{in} and r_{out} = inner and outer radii
Convection	$R_{conv} = \dfrac{1}{\overline{h}\,A_s}$	$\overline{h}$ = average heat transfer coefficient A_s = surface area exposed to convection
Contact between surfaces	$R_c = \dfrac{R_c''}{A_s}$	R_c'' = area specific contact resistance A_s = surface area in contact
Radiation (exact)	$R_{rad} = \dfrac{1}{A_s\,\sigma\,\varepsilon\,(T_s^2 + T_{sur}^2)\,(T_s + T_{sur})}$	A_s = radiating surface area σ = Stefan-Boltzmann constant ε = emissivity T_s = absolute surface temperature T_{sur} = absolute surroundings temperature
Radiation (approximate)	$R_{rad} \approx \dfrac{1}{A_s\,\sigma\,\varepsilon\,4\,\overline{T}^3}$	A_s = radiating surface area σ = Stefan-Boltzmann constant ε = emissivity $\overline{T}$ = average absolute temperature

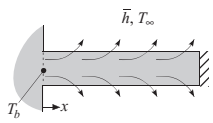
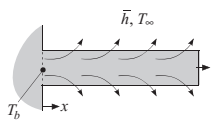
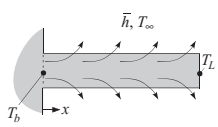
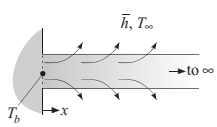
Note that useful reference information, such as Table 1-2, is included in the Heat Transfer Reference Section of EES in order to facilitate solving heat transfer problems without requiring that you locate a written reference book. To access this section, select the Reference Material from the Heat Transfer menu. This will open an online document that contains material from this book. Notice that the Heat Transfer menu also includes all of the examples that are associated with the book.

EXAMPLE 1.2-1: LIQUID OXYGEN DEWAR

Figure 1 illustrates a spherical dewar containing saturated liquid oxygen that is kept at pressure $p_{LOx} = 25$ psia; the saturation temperature of oxygen at this pressure is $T_{LOx} = 95.6$ K.

The dewar consists of an inner and outer metal liner separated by polystyrene foam insulation. The inner metal liner has inner radius $r_{mli,in} = 10.0$ cm and thickness $th_m = 2.5$ mm. The outer metal liner also has thickness $th_m = 2.5$ mm. The conductivity of both metal liners is $k_m = 15$ W/m-K. The heat transfer coefficient between the oxygen within the dewar and the inner surface of the dewar is $\overline{h}_{in} = 150$ W/m^2-K. The outer surface of the dewar is surrounded by air at $T_\infty = 20°$C and radiates to surroundings that are also at $T_\infty = 20°$C. The emissivity of the outer surface of the dewar is $\varepsilon = 0.7$. The heat transfer coefficient between the outer surface of the dewar and the surrounding air is $\overline{h}_{out} = 6$ W/m^2-K. The area-specific

Table 1-4: Solutions for constant cross-section extended surfaces with different end conditions.

	Tip condition	Solution
Adiabatic tip		$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh(m\,(L-x))}{\cosh(m\,L)}$ $\dot{q}_{fin} = (T_b - T_\infty)\sqrt{\bar{h}\,per\,k\,A_c}\,\tanh(m\,L)$ $\eta_{fin} = \tanh(m\,L)\,/\,(m\,L)$
Convection from tip		$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh(m\,(L-x)) + \frac{\bar{h}}{m\,k}\sinh(m\,(L-x))}{\cosh(m\,L) + \frac{\bar{h}}{m\,k}\sinh(m\,L)}$ $\dot{q}_{fin} = (T_b - T_\infty)\sqrt{\bar{h}\,per\,k\,A_c}\,\frac{\sinh(m\,L) + \frac{\bar{h}}{m\,k}\cosh(m\,L)}{\cosh(m\,L) + \frac{\bar{h}}{m\,k}\sinh(m\,L)}$ $\eta_{fin} = \frac{[\tanh(m\,L) + m\,L\,AR_{tip}]}{m\,L\,[1 + m\,L\,AR_{tip}\tanh(m\,L)]\,(1 + AR_{tip})}$
Specified tip temperature		$\frac{T - T_\infty}{T_b - T_\infty} = \frac{\left[\frac{T_L - T_\infty}{T_b - T_\infty}\right]\sinh(m\,x) + \sinh(m\,(L-x))}{\sinh(m\,L)}$ $\dot{q}_{fin} = (T_b - T_\infty)\sqrt{\bar{h}\,per\,k\,A_c}\,\frac{\left(\cosh(m\,L) - \left[\frac{T_L - T_\infty}{T_b - T_\infty}\right]\right)}{\sinh(m\,L)}$
Infinitely long		$\frac{T - T_\infty}{T_b - T_\infty} = \exp(-m\,x)$ $\dot{q}_{fin} = (T_b - T_\infty)\sqrt{\bar{h}\,per\,k\,A_c}$

where:

T_b = base temperature	$\bar{h}$ = heat transfer coefficient
T_∞ = fluid temperature	A_c = cross-sectional area
per = perimeter	k = thermal conductivity
L = length	$\dot{q}_{fin}$ = fin heat transfer rate
T = temperature	x = position (relative to base of fin)
$m\,L = \sqrt{\frac{per\,\bar{h}}{k\,A_c}}\,L$ = fin constant	$AR_{tip} = \frac{A_c}{per\,L}$ = tip area ratio

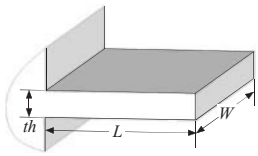
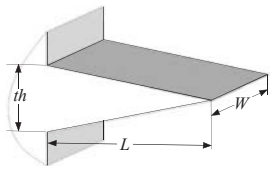
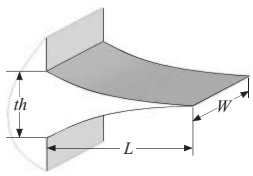
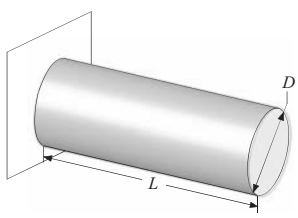
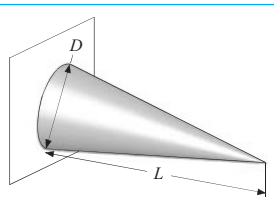
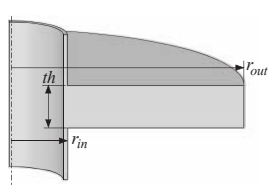
is given by:

$$R_{cond,x} = \frac{L}{k\,A_c} \tag{1-223}$$

and the resistance to convection from the surface (R_{conv}) is:

$$R_{conv} = \frac{1}{\bar{h}\,per\,L} \tag{1-224}$$

Table 1-5: Solutions for extended surfaces.

Shape		Solution
Straight rectangular		$\eta_{fin} = \dfrac{\tanh(m\,L)}{m\,L}$ $A_{s,fin} = 2\,W\,L$ $m\,L = \sqrt{\dfrac{2\,\bar{h}}{k\,th}}\,L$
Straight triangular		$\eta_{fin} = \dfrac{\text{BesselI}(1, 2\,m\,L)}{m\,L\,\text{BesselI}(0, 2\,m\,L)}$ $A_{s,fin} = 2\,W\sqrt{L^2 + \left(\dfrac{th}{2}\right)^2}$ $m\,L = \sqrt{\dfrac{2\,\bar{h}}{k\,th}}\,L$
Straight parabolic		$\eta_{fin} = \dfrac{2}{\left[\sqrt{4\,(m\,L)^2 + 1} + 1\right]}$ $A_{s,fin} = W\left[C_1\,L + \dfrac{L^2}{th}\ln\left(\dfrac{th}{L} + C_1\right)\right]$ $m\,L = \sqrt{\dfrac{2\,\bar{h}}{k\,th}}\,L,\quad C_1 = \sqrt{1 + \left(\dfrac{th}{L}\right)^2}$
Spine rectangular		$\eta_{fin} = \dfrac{\tanh(m\,L)}{m\,L}$ $A_{s,fin} = \pi\,D\,L$ $m\,L = \sqrt{\dfrac{4\,\bar{h}}{k\,D}}\,L$
Spine triangular		$\eta_{fin} = \dfrac{2\text{BesselI}(2, 2\,m\,L)}{m\,L\,\text{BesselI}(1, 2\,m\,L)}$ $A_{s,fin} = \dfrac{\pi\,D}{2}\sqrt{L^2 + \left(\dfrac{D}{2}\right)^2}$ $m\,L = \sqrt{\dfrac{4\,\bar{h}}{k\,D}}\,L$
Rectangular annular		$A_{s,fin} = 2\,\pi\,(r_{out}^2 - r_{in}^2)$ $mr_{out} = \sqrt{\dfrac{2\,\bar{h}}{k\,th}}\,r_{out}$ $mr_{in} = \sqrt{\dfrac{2\,\bar{h}}{k\,th}}\,r_{in}$

$$\eta_{fin} = \frac{2\,mr_{in}}{\left[(mr_{out})^2 - (mr_{in})^2\right]}\,\frac{[\text{BesselK}(1, mr_{in})\ \text{BesselI}(1, mr_{out}) - \text{BesselI}(1, mr_{in})\ \text{BesselK}(1, mr_{out})]}{[\text{BesselI}(0, mr_{in})\ \text{BesselK}(1, mr_{out}) + \text{BesselK}(0, mr_{in})\ \text{BesselI}(1, mr_{out})]}$$

where $\bar{h}$ = heat transfer coefficient k = thermal conductivity

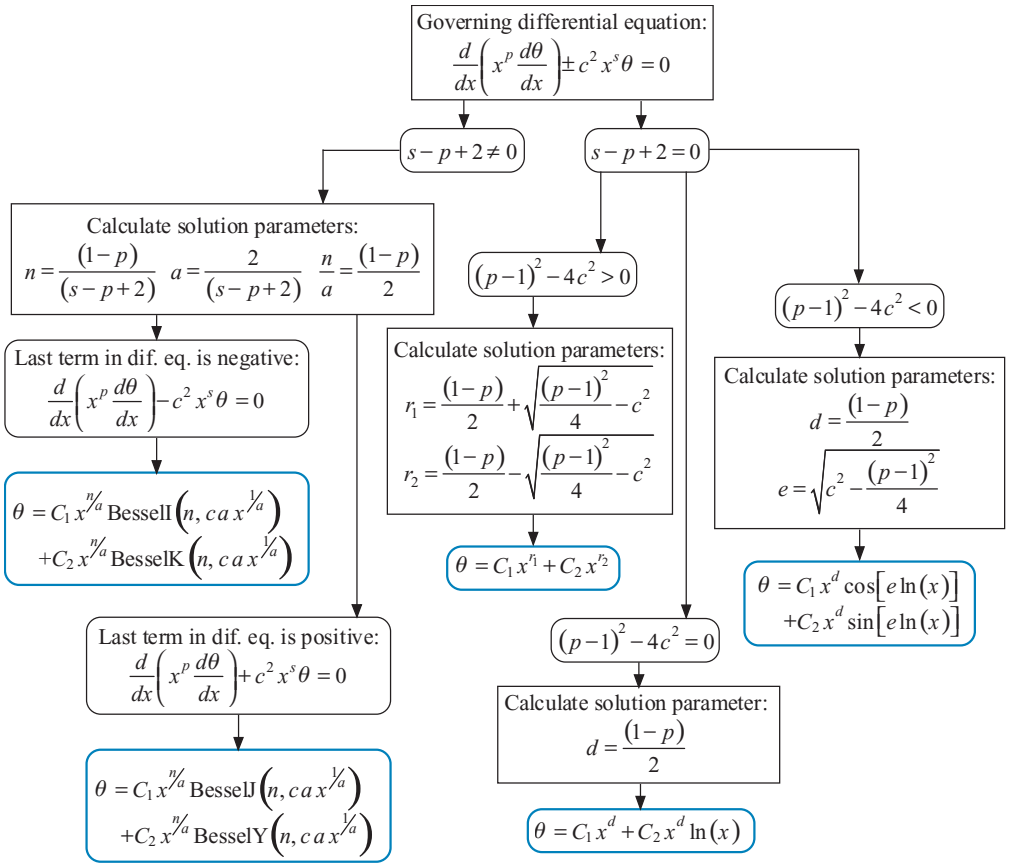


Figure 1-54: Flowchart illustrating the steps involved with identifying the correct solution to Bessel's equation.

where θ is a function of x and p, c, and s are constants is a form of Bessel's equation that has been solved using power series. The rules for identifying the appropriate solution given the form of the equation are laid out in flowchart form in Figure 1-54.

Following the path outlined in Figure 1-54, the first step is to evaluate the quantity $s - p + 2$; if $s - p + 2$ is not equal to zero, then the intermediate solution parameters n and a should be calculated.

$$n = \frac{1-p}{s-p+2} \tag{1-387}$$

$$a = \frac{2}{s-p+2} \tag{1-388}$$

$$\frac{n}{a} = \frac{1-p}{2} \tag{1-389}$$

The solution depends on the sign of the last term in Eq. (1-385); if the sign of the last term is negative, then the solution is expressed as:

$$\theta = C_1 x^{n/a} \text{BesselI}\left(n, c\,a\,x^{1/a}\right) + C_2 x^{n/a} \text{BesselK}\left(n, c\,a\,x^{1/a}\right) \tag{1-390}$$

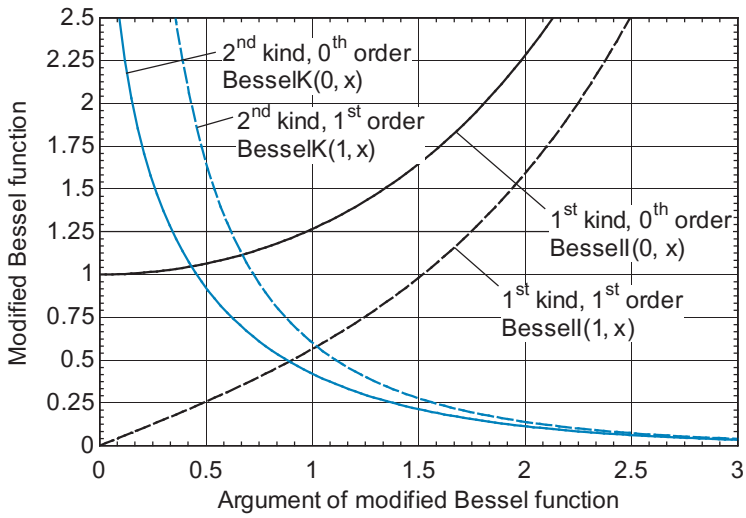


Figure 1-55: Modified Bessel functions of the first and second kinds and the zeroth and first orders.

The zeroth and first order Bessel functions of the first and second kind are shown in Figure 1-56. Notice that the Bessel functions of the second kind, like the modified Bessel functions of the second kind, are unbounded at zero.

The rules for differentiating zeroth order Bessel and zeroth order modified Bessel functions are:

$$\frac{d}{dx}\left[\text{BesselI}(0, u)\right] = \text{BesselI}(1, u)\,\frac{du}{dx} \tag{1-399}$$

$$\frac{d}{dx}\left[\text{BesselK}(0, u)\right] = -\text{BesselK}(1, u)\,\frac{du}{dx} \tag{1-400}$$

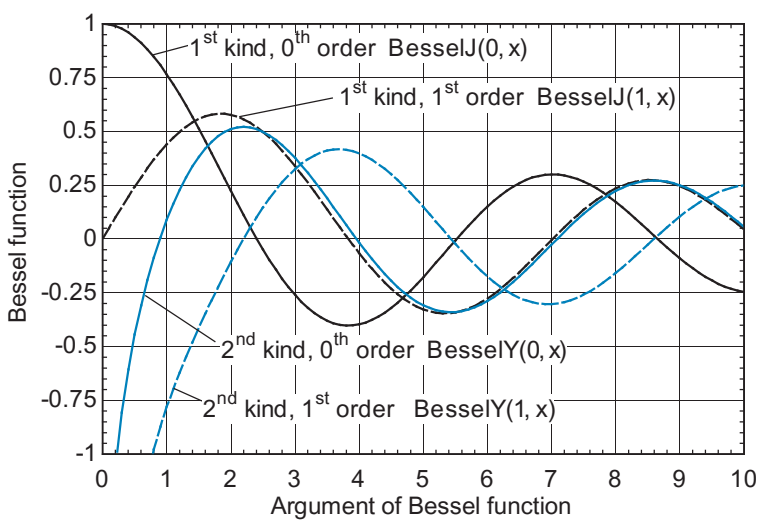


Figure 1-56: Bessel functions of the first and second kinds and the zeroth and first orders.

$$\frac{d}{dx}\left[\text{BesselJ}(0, u)\right] = -\text{BesselJ}(1, u)\,\frac{du}{dx} \tag{1-401}$$

$$\frac{d}{dx}\left[\text{BesselY}(0, u)\right] = -\,\text{BesselY}(1, u)\,\frac{du}{dx} \tag{1-402}$$

For arbitrary order Bessel and modified Bessel functions with positive integer order n, the rules for differentiation are:

$$\frac{d}{dx}\text{BesselI}(n, m\,x) = m\,\text{BesselI}(n-1, m\,x) - \frac{n}{x}\text{BesselI}(n, m\,x) \tag{1-403}$$

$$\frac{d}{dx}\text{BesselK}(n, m\,x) = -m\,\text{BesselK}(n-1, m\,x) - \frac{n}{x}\text{BesselK}(n, m\,x) \tag{1-404}$$

$$\frac{d}{dx}\text{BesselJ}(n, m\,x) = m\,\text{BesselJ}(n-1, m\,x) - \frac{n}{x}\text{BesselJ}(n, m\,x) \tag{1-405}$$

$$\frac{d}{dx}\text{BesselY}(n, m\,x) = m\,\text{BesselY}(n-1, m\,x) - \frac{n}{x}\text{BesselY}(n, m\,x) \tag{1-406}$$

Finally, the following differentials are also sometimes useful:

$$\frac{d}{dx}\left[x^n\,\text{BesselI}(n, m\,x)\right] = m\,x^n\,\text{BesselI}(n-1, m\,x) \tag{1-407}$$

$$\frac{d}{dx}\left[x^n\,\text{BesselK}(n, m\,x)\right] = -m\,x^n\,\text{BesselK}(n-1, m\,x) \tag{1-408}$$

$$\frac{d}{dx}\left[x^n\,\text{BesselJ}(n, m\,x)\right] = m\,x^n\,\text{BesselJ}(n-1, m\,x) \tag{1-409}$$

$$\frac{d}{dx}\left[x^n\,\text{BesselY}(n, m\,x)\right] = -m\,x^n\,\text{BesselY}(n-1, m\,x) \tag{1-410}$$

$$\frac{d}{dx}\left[x^{-n}\,\text{BesselI}(n, m\,x)\right] = m\,x^{-n}\,\text{BesselI}(n+1, m\,x) \tag{1-411}$$

$$\frac{d}{dx}\left[x^{-n}\,\text{BesselK}(n, m\,x)\right] = -m\,x^{-n}\,\text{BesselK}(n+1, m\,x) \tag{1-412}$$

$$\frac{d}{dx}\left[x^{-n}\,\text{BesselJ}(n, m\,x)\right] = -m\,x^{-n}\,\text{BesselJ}(n+1, m\,x) \tag{1-413}$$

$$\frac{d}{dx}\left[x^{-n}\,\text{BesselY}(n, m\,x)\right] = -m\,x^{-n}\,\text{BesselY}(n+1, m\,x) \tag{1-414}$$